NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 50 (2020), 219–231

ON 6-DIMENSIONAL KÄHLER-LIKE OR G-KÄHLER-LIKE NEARLY KÄHLER MANIFOLDS

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(Received 6 January, 2020)

Abstract. We introduce a Kähler-like and a G-Kähler-like almost Hermitian metric. We characterize the Kähler-likeness and the G-Kähler-likeness on nearly Kähler manifolds and prove that a 6-dimensional Kähler-like or G-Kähler-like nearly Kähler manifold is Kähler.

1. Introduction

Yang and Zheng examined the Hermitian curvature tensors of Hermitian metrics, as the curvature tensors satisfies all the symmetry conditions of the curvature tensor of a Kähler metric in [**16**]. They called these metrics Kähler-like and G-Kähler-like. When a manifold is compact, these metrics are more special than balanced metrics since they are always balanced, that is, $d(\omega^{n-1}) = 0$, where ω is the fundamental 2-form associated to a Hermitian metric and n is the complex dimension of the manifold. This fact has attracted attention in the reserch of non-Kähler Calabi-Yau manifolds. Here, recall that balanced metrics play an important role in the Strominger system (cf. [**12**]). From the viewpoint of Mathematics, it is also intriguing to understand the moduli space of Calabi-Yau threefolds (cf. [**13**]). It may be interesting to know if Kähler-like or G-Kähler-like metrics on compact non-Kählerian Calabi-Yau threefolds can play a role in the study of Strominger system or the understanding of the moduli space of Calabi-Yau thereefolds.

Their definitions are as follows. Given a Hermitian manifold (M^n, J, g), there are two canonical connections associated to g, the Chern connection ∇ and the Levi-Civita connection D. Denote R and R^L the curvature tensor of these two connections respectively. Notice that in this whole paper, in the almost Hermitian case M^{2n} indicates that $2n = \dim_{\mathbb{R}} M$, in the Hermitian case M^n means that $n = \dim_{\mathbb{C}} M$.

Definition 1.1. (Kähler-like and G-Kähler-like [**16**]) A Hermitian metric g will be called Kähler-like, if $R_{X\bar{Y}Z\bar{W}} = R_{Z\bar{Y}X\bar{W}}$ holds for any type $(1,0)$ tangent vectors X, Y, Z and W. Similarly, if $R^L_{XY\bar{Z}\bar{W}} = R^L_{XYZ\bar{W}} = 0$ for any type $(1,0)$ tangent vectors X, Y, Z and W, we will say that g is Gray-Kähler-like, or G-Kähler-like for short.

The G-Kähler-like condition was introduced by Gray in [**9**]. Yang and Zheng showed that when $R = R^L$, then g is Kähler in [**16**, Theorem 1.4]. When the manifold is compact, Kähler-like metrics provide important classes of special Hermitian

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2010 *AMS Mathematics Subject Classification:* 32Q60, 53C15, 53C55.
Key words and phrases: almost Hermitian manifolds; Kähler-like metrics; G-Kähler-like metrics; Chern connection.

metrics such as this condition would imply that the metric is balanced (cf. [**16**, Theorem 1.3]). In this sense, the Kähler-likeness and G-Kähler-likeness are more special classes of Hermitian metrics than being balanced for compact Hermitian manifolds. Notice that in the case of $n = 2$, a compact Kähler-like or G-Kähler-like Hermitian surface (M^2, J, g) is Kähler. Vaisman has already showed that any compact G-Kähler-like Hermitian surface is Kähler in [**14**].

We now extend the analysis to almost Hermitian geometry. Let (M, J) be an almost complex manifold and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1, 0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1, 1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1} g_{r\bar{k}} \zeta^r \wedge \zeta^{\bar{k}}$. We will also refer to ω as to an almost Hermitian metric.

We define a Kähler-like almost Hermitian metric in the following as in [**16**]. R^L denotes the curvature tensor with respect to the Levi-Civita connection D on an almost Hermitian manifold. Here, R^L is just the Riemannian curvature tensor, and we extend it linearly over $\mathbb{C}$. R^L is anti-symmetric with respect to their first two or last two positions and symmetric when its first two and last two positions are interchanged, and satisfies the Bianchi identity which means that when one positions is held fixed while the other three are cyclicly permuted, the sum is always zero.

We denote by Ω the curvature of the Chern connection ∇ on an almost Hermitian manifold. We can regard Ω as a section of $\Lambda^2 M \otimes TM$, $\Omega \in \Gamma(\Lambda^2 M \otimes TM)$ and Ω splits in

$$\Omega = \Omega^{(2,0)} + \Omega^{(1,1)} + \Omega^{(0,2)} = H + R + \bar{H},$$

with

$$\Omega^{(2,0)} = \Big(\frac{1}{2} H_{kli}{}^j \zeta^k \wedge \zeta^l\Big), \quad \Omega^{(1,1)} = \Big(R_{k\bar{l}i}{}^j \zeta^k \wedge \zeta^{\bar{l}}\Big), \quad \Omega^{(0,2)} = \Big(\frac{1}{2} H_{\bar{k}\bar{l}i}{}^j \zeta^{\bar{k}} \wedge \zeta^{\bar{l}}\Big).$$

Definition 1.2. Given an almost Hermitian manifold (M^{2n}, J, g), almost Hermitian metric g will be called Kähler-like, if $R_{X\bar{Y}Z\bar{W}} = R_{Z\bar{Y}X\bar{W}}$ for any $(1, 0)$-tangent vectors X, Y, Z and W. When the almost Hermitian metric g is Kähler-like, the triple (M^{2n}, J, g) will be called a Kähler-like almost Hermitian manifold. Similarly, if $R^L_{XY\bar{Z}\bar{W}} = R^L_{XYZ\bar{W}} = 0$ for any type $(1, 0)$ tangent vectors X, Y, Z and W, we will say that g is G-Kähler-like. When the almost Hermitian metric g is G-Kähler-like, the triple (M^{2n}, g, J) will be called a G-Kähler-like almost Hermitian manifold.

When g is Kähler-like, by taking complex conjugations, we see that R is also symmetric with respect to its second and fourth positions, thus obeying all the symmetries of the curvature tensor of a Kähler metric. There are non-Kähler almost Hermitian metrics g which are Kähler-like. For instance, there are almost Hermitian manifolds that are non-Kähler but with $R = 0$ everywhere. Such manifolds are obviously Kähler-like. The Iwasawa manifold is in some fashion (see [**6**, Theorem 4.4]) the unique example of a 6-dimensional non-Kähler almost complex nilmanifold admitting a quasi-Kähler R-flat metric (cf. [**6**], [**16**]).

We investigate Kähler-like or G-Kähler-like nearly Kähler manifolds. Recall that an almost Hermitian manifold (M, J, g) is said to be nearly Kähler if $(D_X J)X = 0$

for any tangent vector field X and $DJ \neq 0$, where D is the Levi-Civita connection associated to g (cf. [**1**, Definition 3.1]). Our main results are as follows.

Proposition 1.3. *Let (M^{2n}, J, g) be a nearly Kähler manifold. Then the Kähler-likeness is equivalent to the G-Kähler-likeness on (M, J, g).*

Theorem 1.4. *Let (M^6, J, g) be a 6-dimensional Kähler-like (or equivalently, G-Kähler-like) nearly Kähler manifold. Then the manifold must be Kähler.*

Since the 6-sphere $\mathbb{S}^6$ admits the strictly (non-Kähler) nearly Kähler structure (J, g) (cf. [**1**]), $(\mathbb{S}^6, J, g)$ is neither Kähler-like nor the G-Kähler-like.

We define the curvature with respect to the Levi-Civita connection D in the following way for tangent vectors X, Y, Z and W:

$$R^L(X, Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z, \quad R^L(X, Y, Z, W) = g(R^L(X, Y)Z, W).$$

An almost Hermitian manifold (M, J, g) satisfying that (cf. [**9**], [**10**])

(1) $R^L(X, Y, Z, W) = R^L(X, Y, JZ, JW)$ for all vector fields X, Y, Z, W is called an AH_1-manifold;

(2) $R^L(X, Y, Z, W) = R^L(X, Y, JZ, JW) + R^L(X, JY, Z, JW) + R^L(JX, Y, Z, JW)$ for all vector fields X, Y, Z, W is called an AH_2-manifold;

(3) $R^L(X, Y, Z, W) = R^L(JX, JY, JZ, JW)$ for all vector fields X, Y, Z, W is called an AH_3-manifold.

(4) $R^L(JX, JY, Z, W) + R^L(JY, JZ, X, W) + R^L(JZ, JX, Y, W) = 0$ for all vector fields X, Y, Z, W is called an AHC-manifold (cf. [**10**]).

Then $AH_1 \subset AH_2 \subset AH_3$, $AH_1 \subset AHC \subset AH_3$, and $AHC \cap AH_2 = AH_1$. Note that if an AH_1-manifold is almost Kähler, then it is Kähler (cf. [**9**, Theorem 5.1]). Furthermore, it is known that an almost Kählerian or a nearly Kählerian AHC-manifold M^{2n} is Kählerian (cf. [**10**, Lemma 10.3]).

The curvature R^L of the Levi-Civita connection D satisfies the first Bianchi identity:

$$\text{(1Bnc)} \quad \sum_{\sigma} R^L(\sigma x, \sigma y)\sigma z = 0.$$

The curvature $\Omega^{(1,1)} = R$ of the Chern connection ∇ satisfies

$$\text{(Cplx)} \quad R(x, y, z, w) = R(x, y, Jz, Jw) = R(Jx, Jy, z, w).$$

We define the Kähler-likeness in the way of [**2**, Definition 4] as follows.

Definition 1.5. Let M be an almost complex manifold endowed with a Hermitian structure. Let ∇ be a metric connection on it. We say that the curvature of ∇ is Kähler-like if it satisfies (1Bnc) and (Cplx).

We see the following equivalences which are similar to the ones in [**2**, Remark 5]. We will give these proofs in the last section as an appendix.

Lemma 1.6. (1) *An almost Hermitian structure is Kähler-like in the sense of Definition 1.2 if and only if the curvature $\Omega^{(1,1)} = R$ of the Chern connection is Kähler-like in the sense of Definition 1.5.*

(2) *An almost Hermitian structure is G-Kähler-like in the sense of Definition 1.2 if and only if the curvature R^L of the Levi-Civita connection is Kähler-like in the sense of Definition 1.5.*

Since we have the symmetry $R^L(x,y,z,w) = R^L(z,w,x,y)$, and since the Levi-Civita connection always satisfies (1Bnc), the Levi-Civita connection being Kähler-like in the sense of Definition 1.5 is equivalent to the AH_1-condition. From Lemma 1.6, consequently, the G-Kähler-condition in Definition 1.2 is equivalent to the AH_1-condition. Combining with the result in Theorem 1.4, we have the following corollary.

Corollary 1.7. *Let (M^6, J, g) be a 6-dimensional AH_1-nearly Kähler manifold. Then the manifold must be Kähler.*

Since $AH_1 \subset AHC$ and that we know that a nearly Kähler AHC-manifold is Kähler, the result of Corollary 1.7 is included in the well-known result.

A quasi-Kähler structure is an almost Hermitian structure whose real $(1,1)$-form ω satisfies $(d\omega)^{(1,2)} = \bar{\partial}\omega = 0$, which is equivalent to the original definition of quasi-Kählerity: $D_X J(Y) + D_{JX} J(JY) = 0$ for all vector fields X, Y (cf. [**8**]). We introduce the following result for 6-dimensional quasi-Kähler manifold.

Proposition 1.8 ([**5**], Main Theorem)**.** *A 6-dimensional quasi-Kähler manifold of constant sectional curvature is a nearly Kähler manifold of positive sectional curvature or a flat Kähler manifold.*

Combining Theorem 1.4 and Proposition 1.8, we have the following corollary.

Corollary 1.9. *A 6-dimentional Kähler-like or G-Kähler-like quasi-Kähler manifold of constant sectional curvature is Kähler.*

We introduce the Iwasawa manifold as in [**6**]. Let G be the 3-dimensional complex Heisenberg group

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C}, i = 1,2,3 \right\}$$

and let Γ be the discrete subgroup of G such that

$$\Gamma := \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{Z} + \sqrt{-1}\mathbb{Z}, i = 1,2,3 \right\}.$$

The subgroup Γ is the co-compact lattice of G and acts on G by left multiplication. Then $M = \Gamma\backslash G$ is a compact 2-step nilpotent nilmanifold, which is called the Iwasawa manifold. A nilmanifold is a quotient $\Gamma\backslash G$ of a connected simply-connected nilpotent Lie group G by a co-compact discrete subgroup Γ. It is well-known that if a nilmanifold $\Gamma\backslash G$ admits a Kähler structure, then G is abelian and $\Gamma\backslash G$ is diffeomorphic to a torus (cf. [**3**, Theorem A]). The Iwasawa manifold admits a global frame $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ which satisfies the structure equations such that

$$[X_1, X_2] = X_3, \quad [X_4, X_5] = -X_3, \quad [X_2, X_4] = X_6, \quad [X_5, X_1] = X_6.$$

The almost complex structure J_0 is defined by

$$\begin{aligned} J_0X_1 &= X_4, & J_0X_2 &= X_5, & J_0X_3 &= X_6, \\ J_0X_4 &= -X_1, & J_0X_5 &= -X_2, \text{ and} & J_0X_6 &= -X_3. \end{aligned}$$

Let g_0 be the J_0-almost Hermitian metric $g_0 = \sum_{i=1}^{6} \alpha_i \otimes \alpha_i$ and let $\omega_0 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6$, where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be the coframe associated to $\{X_1, X_2, X_3, X_4, X_5, X_6\}$. Then (J_0, g_0, ω_0) is a quasi-Kähler structure on M. Since the curvature tensor of the Chern connection associated to (J_0, g_0) vanishes (cf. [6. Theorem 4.3]), it means that (M, J_0, g_0) is automatically Kähler-like. If the Iwasawa manifold (M, J_0, g_0) admits a constant sectional curvature, then from Corollary 1.9, it must be Kähler, which contradicts to that the Iwasawa manifold is non-Kähler. This implies that the Iwasawa manifold (M, J_0, g_0) does not admit constant sectional curvature.

This paper is organized as follows: in section 2, we recall some basic definitions and computations. In section 3, we characterize the Kähler-likeness and the G-Kähler-likeness on nearly Kähler manifolds and prove that a Kähler-like or G-Kähler-like nearly Kähler manifold is Kähler. In the last section, we will give a proof of Lemma 1.6 for readers' convenience. Notice that we assume the Einstein convention omitting the symbol of sum over repeated indexes in all this paper.

2. Preliminaries

2.1. The Nijenhuis tensor of the almost complex structure. Let M be a $2n$-dimensional smooth differentiable manifold. An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -Id_{TM}$. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. We define a bilinear map on $C^\infty(M)$ for $X, Y \in \Gamma(TM)$ by

$$4N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

which is the Nijenhuis tensor of J. The Nijenhuis tensor N satisfies

$$N(X, Y) = -N(Y, X) \qquad N(JX, Y) = -JN(X, Y)$$
$$N(X, JY) = -JN(X, Y) \qquad N(JX, JY) = -N(X, Y).$$

For any $(1, 0)$-vector fields W and V we know that $N(V, W) = -[V, W]^{(0,1)}$, $N(V, \bar{W}) = N(\bar{V}, W) = 0$ and $N(\bar{V}, \bar{W}) = -[\bar{V}, \bar{W}]^{(1,0)}$ since it holds that $4N(V, W) = -2([V, W] + \sqrt{-1}J[V, W])$, $4N(\bar{V}, \bar{W}) = -2([\bar{V}, \bar{W}] - \sqrt{-1}J[\bar{V}, \bar{W}])$. An almost complex structure J is called integrable if $N = 0$ everywhere on M. Giving a complex structure on a differentiable manifold M is equivalent to giving an integrable almost complex structure on M. A Riemannian metric g on M is called J-invariant if J is compatible with g, i.e., for any $X, Y \in \Gamma(TM)$, $g(X, Y) = g(JX, JY)$. In this case, the pair (J, g) is called an almost Hermitian structure. The fundamental 2-form ω associated to a J-invariant Riemannian metric g, i.e., an almost Hermitian metric, is determined by, for $X, Y \in \Gamma(TM)$, $\omega(X, Y) = g(JX, Y)$. Indeed we have, for any $X, Y \in \Gamma(TM)$,

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y)$$

and $\omega \in \Gamma(\bigwedge^2 T^*M)$. We will also refer to the associated real fundamental $(1, 1)$-form ω as an almost Hermitian metric. The form ω is related to the volume form dV_g by $n!dV_g = \omega^n$. Let a local $(1, 0)$-frame $\{Z_r\}$ on (M, J) with an almost Hermitian metric g and let $\{\zeta^r\}$ be a local associated coframe with respect to $\{Z_r\}$, i.e., $\zeta^i(Z_j) = \delta^i_j$ for $i, j = 1, \ldots, n$. Since g is almost Hermitian, its components satisfy $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i} = \bar{g}_{\bar{i}j}$.

By using these local frame $\{Z_r\}$ and coframe $\{\zeta^r\}$, we have

$$N(Z_{\bar{i}}, Z_{\bar{j}}) = -[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)} =: N_{\bar{i}\bar{j}}^{k} Z_k, \quad N(Z_i, Z_j) = -[Z_i, Z_j]^{(0,1)} = \overline{N_{\bar{i}\bar{j}}^{k}} Z_{\bar{k}},$$

$$N = \frac{1}{2}\overline{N_{\bar{i}\bar{j}}^{k}} Z_{\bar{k}} \otimes (\zeta^i \wedge \zeta^j) + \frac{1}{2} N_{\bar{i}\bar{j}}^{k} Z_k \otimes (\zeta^{\bar{i}} \wedge \zeta^{\bar{j}}).$$

We write $T^{\mathbb{R}}M$ for the real tangent space of M. Then its complexified tangent space is given by $T^{\mathbb{C}}M = T^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$. By extending J $\mathbb{C}$-linearly and g, ω, $\mathbb{C}$-bilinearly to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent space $T^{\mathbb{C}}M$ can be decomposed as $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$, $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively:

$$T^{1,0}M = \{X - \sqrt{-1}JX \big| X \in TM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \big| X \in TM\}.$$

Let $\Lambda^r M = \bigoplus_{p+q=r} \Lambda^{p,q}M$ for $0 \le r \le 2n$ denote the decomposition of complex differential r-forms into (p,q)-forms, where $\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M)$,

$$\Lambda^{1,0}M = \{\eta + \sqrt{-1}J\eta \big| \eta \in \Lambda^1 M\}, \quad \Lambda^{0,1}M = \{\eta - \sqrt{-1}J\eta \big| \eta \in \Lambda^1 M\}$$

and $\Lambda^1 M$ denotes the dual of TM.

Let (M^{2n}, J, g) be an almost Hermitian manifold. An affine connection D on TM is called almost Hermitian connection if $Dg = DJ = 0$. For the almost Hermitian connection, we have the following Lemma (cf. [**4**], [**7**], [**15**], [**19**]).

Lemma 2.1. *Let (M, g, J) be an almost Hermitian manifold with $\dim_{\mathbb{R}} M = 2n$. Then for any given vector valued $(1,1)$-form $\Theta = (\Theta^i)_{1\le i\le n}$, there exists a unique almost Hermitian connection D on (M, J, g) such that the $(1,1)$-part of the torsion is equal to the given Θ.*

If the $(1,1)$-part of the torsion of an almost Hermitian connection vanishes everywhere, then the connction is called the second canonical connection or the Chern connection. We will refer the connection as the Chern connection and denote it by ∇.

Now let ∇ be the Chern connection on M. We denote the structure coefficients of Lie bracket by

$$[Z_i, Z_j] =: B_{ij}^{r} Z_r + B_{ij}^{\bar{r}} Z_{\bar{r}} = B_{ij}^{r} Z_r - \overline{N_{\bar{i}\bar{j}}^{r}} Z_{\bar{r}},, \quad [Z_i, Z_{\bar{j}}] =: B_{i\bar{j}}^{r} Z_r + B_{i\bar{j}}^{\bar{r}} Z_{\bar{r}},$$

$$[Z_{\bar{i}}, Z_{\bar{j}}] =: B_{\bar{i}\bar{j}}^{r} Z_r + B_{\bar{i}\bar{j}}^{\bar{r}} Z_{\bar{r}} = -N_{\bar{i}\bar{j}}^{r} Z_r + B_{\bar{i}\bar{j}}^{\bar{r}} Z_{\bar{r}},$$

where we used that $[Z_i, Z_j]^{(0,1)} = -\overline{N_{\bar{i}\bar{j}}^{r}} Z_{\bar{r}}$, $[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)} = -N_{\bar{i}\bar{j}}^{r} Z_r$ and then $B_{ij}^{\bar{r}} = -\overline{N_{\bar{i}\bar{j}}^{r}}$, $B_{\bar{i}\bar{j}}^{r} = -N_{\bar{i}\bar{j}}^{r}$. Also we here note that for instance, $[Z_i, Z_{\bar{j}}] = [Z_i, Z_{\bar{j}}]^{(1,0)} + [Z_i, Z_{\bar{j}}]^{(0,1)}$, where

$$[Z_i, Z_{\bar{j}}]^{(1,0)} = \frac{1}{2}([Z_i, Z_{\bar{j}}] - \sqrt{-1}J[Z_i, Z_{\bar{j}}]), \quad [Z_i, Z_{\bar{j}}]^{(0,1)} = \frac{1}{2}([Z_i, Z_{\bar{j}}] + \sqrt{-1}J[Z_i, Z_{\bar{j}}]).$$

Notice that J is integrable if and only if the $B_{ij}^{\bar{r}}$'s vanish.

2.2. The torsion and the curvature on almost complex manifolds. Since the Chern connection ∇ preserves J, we are able to define the Christoffel symbols: for $i,j,r=1,\dots,n$,

$$\nabla_i Z_j = \nabla_{Z_i} Z_j = \Gamma^r_{ij} Z_r, \quad \nabla_i Z_{\bar j} = \nabla_{Z_i} Z_{\bar j} = \Gamma^{\bar r}_{i\bar j} Z_{\bar r},$$

where

$$\Gamma^r_{ij} = g^{r\bar s} Z_i(g_{j\bar s}) - g^{r\bar s} g_{j\bar l} B^{\bar l}_{i\bar s}, \quad \Gamma^p_{ip} = Z_i(\log\det g) - B^{\bar s}_{i\bar s}.$$

The torsion $T=(T^i)$ of the Chern connection ∇ is defined by

$$T^i := d\zeta^i - \zeta^p \wedge \omega^i_p, \quad T^{\bar i} := d\zeta^{\bar i} - \zeta^{\bar p} \wedge \omega^{\bar i}_{\bar p},$$

where $\omega = (\omega^i_j)$ is the connection form defined by $\omega^i_j := \Gamma^i_{kj}\zeta^k + \Gamma^i_{\bar k j}\zeta^{\bar k}$. Since the torsion T of the Chern connection ∇ has no $(1,1)$-part;

$$0 = T^i_{k\bar l} = T^i(Z_k, Z_{\bar l}) = -\zeta^i([Z_k, Z_{\bar l}]) - (\Gamma^i_{sp}\zeta^p\wedge\zeta^s + \Gamma^i_{\bar s p}\zeta^p\wedge\zeta^{\bar s})(Z_k, Z_{\bar l}) = -B^i_{k\bar l} - \Gamma^i_{\bar l k}$$

$$0 = T^{\bar i}_{k\bar l} = T^{\bar i}(Z_k, Z_{\bar l}) = -\zeta^{\bar i}([Z_k, Z_{\bar l}]) - (\Gamma^{\bar i}_{s\bar p}\zeta^{\bar p}\wedge\zeta^s + \Gamma^{\bar i}_{\bar s\bar p}\zeta^{\bar p}\wedge\zeta^{\bar s})(Z_k, Z_{\bar l}) = -B^{\bar i}_{k\bar l} + \Gamma^{\bar i}_{k\bar l},$$

we have

$$\Gamma^{\bar r}_{i\bar j} = B^{\bar r}_{i\bar j}.$$

Here note that $B^{\bar q}_{j\bar b}$, $B^q_{\bar j b}$'s do not depend on g, which depend only on J since the mixed derivatives $\nabla_j Z_{\bar b}$ do not depend on g (cf. [**15**]).

The torsion T of ∇ has no $(1,1)$-part and the only non-vanishing components are as follows:

$$T^s_{ij} = T^s(Z_i, Z_j) = -\zeta^s([Z_i, Z_j]) - (\Gamma^i_{sp}\zeta^p\wedge\zeta^s + \Gamma^i_{\bar s p}\zeta^p\wedge\zeta^{\bar s})(Z_i, Z_j) = -B^s_{ij} - \Gamma^s_{ji} + \Gamma^s_{ij},$$

$$T^{\bar s}_{ij} = d\zeta^{\bar s}(Z_i, Z_j) = -\zeta^{\bar s}([Z_i, Z_j]) = -B^{\bar s}_{ij}$$

and on the other hand we have $d\zeta^{\bar s}(Z_i, Z_j) = N^{\bar s}_{ij}$, hence we obtain that $T^{\bar s}_{ij} = N^{\bar s}_{ij} = -B^{\bar s}_{ij}$. These computations tell us that T splits in $T = T' + T''$, where $T' \in \Gamma(\Lambda^{2,0}M \otimes T^{1,0}M)$, a section of $\Lambda^{2,0}M\otimes T^{1,0}M$, and $T'' \in \Gamma(\Lambda^{2,0}M\otimes T^{0,1}M)$. The torsion $T=(T^i)$ can be split into $T = T^{2,0} + T^{1,1} + T^{0,2} = T^{2,0} + T^{0,2}$ since $T^{1,1} = 0$, where $T^{2,0} = \left(\frac{1}{2}T^i_{jk}\zeta^j\wedge\zeta^k\right)_{1\le i\le n}$, $T^{0,2} = \left(\frac{1}{2}N^i_{\bar j\bar k}\zeta^{\bar j}\wedge\zeta^{\bar k}\right)_{1\le i\le n}$, which tells us that $(0,2)$-part of the Chern connection is uniquely determined by the Nijenhuis tensor (cf. [**19**]).

We also lower the index of torsion and denote it by $T_{ij\bar k} = T^s_{ij} g_{s\bar k}$. Note that T'' depends only on J and it can be regarded as the Nijenhuis tensor of J, that is, J is integrable if and only if T'' vanishes.

We denote by Ω the curvature of the Chern connection ∇. We can regard Ω as a section of $\Lambda^2 M \otimes TM$, and Ω splits in $\Omega = H + R + \bar H$ with

$$\Omega^{(2,0)} = \left(\frac{1}{2}H_{kli}{}^j\zeta^k\wedge\zeta^l\right), \quad \Omega^{(1,1)} = \left(R_{k\bar l i}{}^j\zeta^k\wedge\zeta^{\bar l}\right), \quad \Omega^{(0,2)} = \left(\frac{1}{2}H_{\bar k\bar l i}{}^j\zeta^{\bar k}\wedge\zeta^{\bar l}\right),$$

where $R \in \Gamma(\Lambda^{1,1}M \otimes \Lambda^{1,1}M)$, $H \in \Gamma(\Lambda^{2,0}M \otimes \Lambda^{1,1}M)$. The curvature form can be written by $\Omega^i_j = d\omega^i_j + \omega^i_s \wedge \omega^s_j$.

In terms of Z_r's, we have

$$R_{i\bar j k}{}^r = \Omega^r_k(Z_i, Z_{\bar j}) = Z_i(\Gamma^r_{\bar j k}) - Z_{\bar j}(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{\bar j k} - \Gamma^r_{\bar j s}\Gamma^s_{ik} - B^s_{i\bar j}\Gamma^r_{sk} + B^{\bar s}_{\bar j i}\Gamma^r_{\bar s k},$$

$$H_{ijk}{}^r = \Omega^r_k(Z_i, Z_j) = Z_i(\Gamma^r_{jk}) - Z_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk} - B^{\bar s}_{ij}\Gamma^r_{\bar s k},$$

$$H_{\bar i\bar j k}{}^r = \Omega^r_k(Z_{\bar i}, Z_{\bar j}) = Z_{\bar i}(\Gamma^r_{\bar j k}) - Z_{\bar j}(\Gamma^r_{\bar i k}) + \Gamma^r_{\bar i s}\Gamma^s_{\bar j k} - \Gamma^r_{\bar j s}\Gamma^s_{\bar i k} - B^s_{\bar i\bar j}\Gamma^r_{sk} - B^{\bar s}_{\bar i\bar j}\Gamma^r_{\bar s k},$$

and we deduce that with using $\Gamma^p_{kp} = Z_k(\log\det g) - B^{\bar{p}}_{k\bar{p}}$,

$$P_{i\bar{j}} = R_{i\bar{j}r}{}^r = -(Z_iZ_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})(\log\det g) + Z_{\bar{j}}(B^{\bar{r}}_{i\bar{r}}) + Z_i(B^r_{\bar{j}r}) + B^s_{i\bar{j}}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{i\bar{j}}B^r_{\bar{s}r}$$

$$R_{ij} = H_{ijr}{}^r = [Z_i, Z_j]^{(0,1)}(\log\det g) - Z_i(B^{\bar{r}}_{j\bar{r}}) + Z_j(B^{\bar{r}}_{i\bar{r}}) + B^s_{ij}B^{\bar{r}}_{s\bar{r}} + \overline{N^{\bar{s}}_{\bar{i}\bar{j}}}B^r_{\bar{s}r},$$

$$R_{\bar{i}\bar{j}} = H_{\bar{i}\bar{j}r}{}^r = -[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)}(\log\det g) + Z_{\bar{i}}(B^r_{\bar{j}r}) - Z_{\bar{j}}(B^r_{\bar{i}r}) - N^s_{\bar{i}\bar{j}}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{\bar{i}\bar{j}}B^r_{\bar{s}r}.$$

The curvature P is one of the Ricci-type curvatures of the Chern curvature. One has with an arbitrary $(1,0)$-frame $\{Z_r\}$ with respect to g, $P_{i\bar{j}} = g^{k\bar{l}}\Omega_{i\bar{j}k\bar{l}}$. We denote by S one of the Ricci-type curvatures of the Chern curvature, which is locally given by $S_{i\bar{j}} = g^{k\bar{l}}\Omega_{k\bar{l}i\bar{j}}$. In the Kähler case, $S_{i\bar{j}} = P_{i\bar{j}}$ is the Ricci curvature.

2.3. Complexified Riemannian curvature tensor. We introduce the complexified Riemannian curvature tensor as in [**11**]. Let (M, g, D) be a $2n$-dimensional Riemannian manifold with Levi-Civita connection D. The tangent bundle of M is denoted by $T^{\mathbb{R}}M$. The curvature tensor of (M, g, D) is defined as

$$R^L(X, Y, Z, W) = g(D_ZD_WX - D_WD_ZX - D_{[Z,W]}X, Y)$$

for any $X, Y, Z, W \in T^{\mathbb{R}}M$. Let $T^{\mathbb{C}}M = T^{\mathbb{R}}M \otimes \mathbb{C}$ be the complexification of the tangent bundle $T^{\mathbb{R}}M$. We can extend the metric g, the Levi-Civita connection D to $T^{\mathbb{C}}M$ in the $\mathbb{C}$-linear way. For any $a, b \in \mathbb{C}$ and $X, Y \in T^{\mathbb{C}}M$, $g(aX, bY) := ab \cdot g(X, Y)$. Hence for any $a, b, c, d \in \mathbb{C}$ and $X, Y, Z, W \in T^{\mathbb{C}}M$, $R^L(aX, bY, cZ, dW) = abcd \cdot R^L(X, Y, Z, W)$. Notice that the components of $\mathbb{C}$-linear complexified curvature tensor have the same properties as the components of the real curvature tensor. The components of the complexified curvature tensor $R_{i\bar{j}k\bar{l}}$ satisfies the following properties: $R^L_{i\bar{j}k\bar{l}} = -R^L_{\bar{j}ik\bar{l}}$, $R^L_{i\bar{j}k\bar{l}} = R^L_{k\bar{l}i\bar{j}}$ and the Bianchi identity: $R^L_{i\bar{j}k\bar{l}} + R^L_{ik\bar{l}\bar{j}} + R^L_{i\bar{l}\bar{j}k} = 0$.

3. Proof of Theorem 1.4

First, we show the following identity of the Chern curvature.

Lemma 3.1. *Let (M, J, g) be an almost Hermitian manifold and let ∇ be the Chern connection with respect to g. For a local unitary $(1,0)$-frame $\{Z_r\}$ with respect to g, then one has*

$$R_{i\bar{j}k\bar{l}} = g(\nabla_i\nabla_{\bar{j}}Z_k - \nabla_{\bar{j}}\nabla_iZ_k - \nabla_{[Z_i,Z_{\bar{j}}]}Z_k, Z_{\bar{l}}).$$

Proof. By using a local g-unitary $(1,0)$-frame $\{Z_r\}$, we obtain the following:

$$\begin{aligned}
& g(\nabla_i\nabla_{\bar{j}}Z_k - \nabla_{\bar{j}}\nabla_iZ_k - \nabla_{[Z_i,Z_{\bar{j}}]}Z_k, Z_{\bar{l}}) \\
= \; & Z_i(g(\nabla_{\bar{j}}Z_k, Z_{\bar{l}})) - g(\nabla_{\bar{j}}Z_k, \nabla_iZ_{\bar{l}}) - Z_{\bar{j}}(g(\nabla_iZ_k, Z_{\bar{l}}) + g(\nabla_iZ_k, \nabla_{\bar{j}}Z_{\bar{l}}) \\
& -B^r_{i\bar{j}}g(\nabla_rZ_k, Z_{\bar{l}}) - B^{\bar{r}}_{i\bar{j}}g(\nabla_{\bar{r}}Z_k, Z_{\bar{l}}) \\
= \; & Z_i(\Gamma^s_{\bar{j}k}g_{s\bar{l}}) - \Gamma^s_{\bar{j}k}\Gamma^{\bar{r}}_{i\bar{l}}g_{s\bar{r}} - Z_{\bar{j}}(\Gamma^s_{ik}g_{s\bar{l}}) + \Gamma^s_{ik}\Gamma^{\bar{r}}_{\bar{j}\bar{l}}g_{s\bar{r}} - B^s_{i\bar{j}}\Gamma^r_{sk}g_{r\bar{l}} - B^{\bar{r}}_{i\bar{j}}\Gamma^s_{\bar{r}k}g_{s\bar{l}} \\
= \; & Z_i(\Gamma^l_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^l_{ik}) + \Gamma^l_{is}\Gamma^s_{\bar{j}k} - \Gamma^l_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^l_{sk} - B^{\bar{s}}_{i\bar{j}}\Gamma^l_{\bar{s}k} \\
= \; & R_{i\bar{j}k}{}^sg_{s\bar{l}} \\
= \; & R_{i\bar{j}k\bar{l}},
\end{aligned}$$

where we have used that $\Gamma^k_{ij} = -\Gamma^{\bar{j}}_{i\bar{k}}$. Therefore, we obtain the desired formula. $\square$

Let (M^{2n}, J, g) be an almost Hermitian manifold and let ∇ be the Chern connection on M. Let $\{Z_r\}$ be a local $(1,0)$-frame with respect to g around a fixed point $p \in M$ such that $\nabla Z_i(p) = 0$ and $g_{i\bar{j}}(p) = \delta_{ij}$ and let $\{\zeta^r\}$ be the associated coframe. The existence of such frames has been proven in [**18**].

We introduce the following equivalence for nearly Kähler manifolds.

Lemma 3.2 ([**17**] Lemma 2.4)**.** *An almost Hermitian manifold (M, J, g) is nearly Kähler if and only if $T^k_{ij} = 0$ and $T^{\bar{k}}_{ij} = T^{\bar{i}}_{jk}$ for all i, j and k when a local unitary $(1,0)$-frame is fixed.*

Lemma 3.3 ([**17**] Theorem 2.1)**.** *Let (M, J, g) is a nearly Kähler manifold. Then $\nabla T = 0$.*

Lemma 3.4 ([**17**] Corollary 3.8)**.** *Let (M, J, g) be a nearly Kähler manifold and fix a unitary frame. Then*

$$R^L_{ijk\bar{l}} = 0.$$

Now we give a proof of Proposition 1.3.

Proof of Proposition 1.3. Suppose that (M^{2n}, J, g) be a nearly Kähler manifold. Let D be the Levi-Civita connection and R^L denotes its curvature tensor with respect to g. Recall the definition of curvature operator: for type $(1,0)$ tangent vectors X, Y, Z,

$$R^L(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z.$$

The curvature tensor is defined for type $(1,0)$ tangent vectors X, Y, Z and W as

$$R^L(X,Y,Z,W) = g(R^L(Z,W)X, Y).$$

The torsion T with respect to the Chern connection ∇ is a vector-valued 2-form given by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ satisfying $T(X,\bar{Y}) = 0$ for any type $(1,0)$ tangent vectors X, Y. Recall the following comparison of the Levi-Civita connection and the Chern connection on almost Hermitian manifolds.

Lemma 3.5 (cf. [**17**] Lemma 3.1)**.** *It holds that*

$$g(D_Y X, Z) = g(\nabla_Y X, Z) + \frac{1}{2}(g(T(X,Y),Z) + g(T(Y,Z),X) - g(T(Z,X),Y))$$

for any type $(1,0)$ tangent vectors X, Y, Z.

For an arbitrary chosen fixed point $p \in M$, let $\{Z_r\}$ be a local $(1,0)$-frame at p such that $\nabla Z_r(p) = 0$ and $g_{i\bar{j}}(p) = \delta_{ij}$. By Lemma 3.5, we compute at p:

$$\begin{aligned}
g(D_{Z_{\bar{\lambda}}} Z_i, Z_\mu) &= -\frac{1}{2} g(T(Z_\mu, Z_i), Z_{\bar{\lambda}}) = -\frac{1}{2} T^\lambda_{\mu i} = \frac{1}{2} T^\lambda_{i\mu},\\
g(D_{Z_{\bar{\lambda}}} Z_i, Z_{\bar{\mu}}) &= \frac{1}{2} g(T(Z_{\bar{\lambda}}, Z_{\bar{\mu}}), Z_i) = \frac{1}{2} T^{\bar{i}}_{\bar{\lambda}\bar{\mu}},\\
g(D_\lambda Z_i, Z_{\bar{\mu}}) &= \frac{1}{2} g(T(Z_i, Z_\lambda), Z_{\bar{\mu}}) = \frac{1}{2} T^\mu_{i\lambda},\\
g(D_\lambda Z_i, Z_\mu) &= \frac{1}{2}(g(T(Z_i, Z_\lambda), Z_\mu) + g(T(Z_\lambda, Z_\mu), Z_i) - g(T(Z_\mu, Z_i), Z_\lambda))\\
&= \frac{1}{2}(T^{\bar{\mu}}_{i\lambda} + T^{\bar{i}}_{\lambda\mu} - T^{\bar{\lambda}}_{\mu i}).
\end{aligned}$$

Hence, we obtain

$$D_{Z_{\bar\lambda}} Z_i(p) = \frac{1}{2} T^{\lambda}_{i\mu}(p) Z_{\bar\mu} + \frac{1}{2} T^{\bar i}_{\bar\lambda\bar\mu}(p) Z_{\mu}$$

and

$$D_{Z_{\lambda}} Z_i(p) = \frac{1}{2} T^{\mu}_{i\lambda}(p) Z_{\mu} + \frac{1}{2}(T^{\bar\mu}_{i\lambda} + T^{\bar i}_{\lambda\mu} - T^{\bar\lambda}_{\mu i})(p) Z_{\bar\mu}.$$

Using Lemma 3.5 and the two equations just above, we compute at p,

$$\begin{aligned}
g(D_{Z_{\bar k}} D_{Z_{\bar l}} Z_i, Z_j) &= g(\nabla_{Z_{\bar k}} D_{Z_{\bar l}} Z_i, Z_j) + \frac{1}{2}(g(T(D_{Z_{\bar l}} Z_i, Z_{\bar k}), Z_j) - g(T(Z_j, D_{Z_{\bar l}} Z_i), Z_{\bar k})) \\
&= Z_{\bar k}(g(D_{Z_{\bar l}} Z_i, Z_j)) + \frac{1}{4} T^{l}_{i\lambda} T^{\bar j}_{\bar\lambda\bar k} - \frac{1}{4} T^{\bar i}_{\bar l\bar\lambda} T^{k}_{j\lambda} \\
&= Z_{\bar k}(g(\nabla_{\bar l} Z_i, Z_j) - \frac{1}{2} g(T(Z_j, Z_i), Z_{\bar l})) + \frac{1}{4} T^{l}_{i\lambda} T^{\bar j}_{\bar\lambda\bar k} - \frac{1}{4} T^{\bar i}_{\bar l\bar\lambda} T^{k}_{j\lambda} \\
&= g(\nabla_{\bar k}\nabla_{\bar l} Z_i, Z_j) - \frac{1}{2}\nabla_{\bar k} T^{l}_{ji} + \frac{1}{4} T^{l}_{i\lambda} T^{\bar j}_{\bar\lambda\bar k} - \frac{1}{4} T^{\bar i}_{\bar l\bar\lambda} T^{k}_{j\lambda}.
\end{aligned}$$

Similarly, we have at p,

$$g(D_{Z_{\bar l}} D_{Z_{\bar k}} Z_i, Z_j) = g(\nabla_{\bar l}\nabla_{\bar k} Z_i, Z_j) - \frac{1}{2}\nabla_{\bar l} T^{k}_{ji} + \frac{1}{4} T^{k}_{i\lambda} T^{\bar j}_{\bar\lambda\bar l} - \frac{1}{4} T^{\bar i}_{\bar k\bar\lambda} T^{l}_{j\lambda}.$$

Since we have

$$\begin{aligned}
[Z_{\bar k}, Z_{\bar l}](p) &= \nabla_{\bar k} Z_{\bar l}(p) - \nabla_{\bar l} Z_{\bar k}(p) - T(Z_{\bar k}, Z_{\bar l})(p) \\
&= -T^{\lambda}_{\bar k\bar l}(p) Z_{\lambda} - T^{\bar\lambda}_{\bar k\bar l}(p) Z_{\bar\lambda},
\end{aligned}$$

we then get at p,

$$\begin{aligned}
g(D_{[Z_{\bar k}, Z_{\bar l}]} Z_i, Z_j) &= -T^{\lambda}_{\bar k\bar l} g(D_{Z_\lambda} Z_i, Z_j) - T^{\bar\lambda}_{\bar k\bar l} g(D_{Z_{\bar\lambda}} Z_i, Z_j) \\
&= -\frac{1}{2} T^{\lambda}_{\bar k\bar l}(T^{\bar j}_{i\lambda} + T^{\bar i}_{\lambda j} - T^{\bar\lambda}_{ji}) - \frac{1}{2} T^{\bar\lambda}_{\bar k\bar l} T^{\lambda}_{ij}.
\end{aligned}$$

Hence, we have at p,

$$\begin{aligned}
(\dagger)\quad R^{L}_{ij\bar k\bar l} &= g(D_{Z_{\bar k}} D_{Z_{\bar l}} Z_i - D_{Z_{\bar l}} D_{Z_{\bar k}} Z_i - D_{[Z_{\bar k}, Z_{\bar l}]} Z_i, Z_j) \\
&= g((\nabla_{\bar k}\nabla_{\bar l} - \nabla_{\bar l}\nabla_{\bar k} - \nabla_{[Z_{\bar k}, Z_{\bar l}]}) Z_i, Z_j) - \frac{1}{2}\nabla_{\bar k} T^{l}_{ji} + \frac{1}{2}\nabla_{\bar l} T^{k}_{ji} + \frac{1}{4} T^{l}_{i\lambda} T^{\bar j}_{\bar\lambda\bar k} \\
&\quad - \frac{1}{4} T^{\bar i}_{\bar l\bar\lambda} T^{k}_{j\lambda} - \frac{1}{4} T^{k}_{i\lambda} T^{\bar j}_{\bar\lambda\bar l} + \frac{1}{4} T^{\bar i}_{\bar k\bar\lambda} T^{l}_{j\lambda} + \frac{1}{2} T^{\lambda}_{\bar k\bar l}(T^{\bar j}_{i\lambda} + T^{\bar i}_{\lambda j} - T^{\bar\lambda}_{ji}) + \frac{1}{2} T^{\bar\lambda}_{\bar k\bar l} T^{\lambda}_{ij} \\
&= R_{\bar k\bar l ij} - \frac{1}{2}\nabla_{\bar k} T^{l}_{ji} + \frac{1}{2}\nabla_{\bar l} T^{k}_{ji} + \frac{1}{4} T^{l}_{i\lambda} T^{\bar j}_{\bar\lambda\bar k} - \frac{1}{4} T^{\bar i}_{\bar l\bar\lambda} T^{k}_{j\lambda} - \frac{1}{4} T^{k}_{i\lambda} T^{\bar j}_{\bar\lambda\bar l} + \frac{1}{4} T^{\bar i}_{\bar k\bar\lambda} T^{l}_{j\lambda} \\
&\quad + \frac{1}{2} T^{\lambda}_{\bar k\bar l}(T^{\bar j}_{i\lambda} + T^{\bar i}_{\lambda j} - T^{\bar\lambda}_{ji}) + \frac{1}{2} T^{\bar\lambda}_{\bar k\bar l} T^{\lambda}_{ij} \\
&= -\frac{1}{2}\nabla_{\bar k} T^{l}_{ji} + \frac{1}{2}\nabla_{\bar l} T^{k}_{ji} + \frac{1}{4} T^{l}_{i\lambda} T^{\bar j}_{\bar\lambda\bar k} - \frac{1}{4} T^{\bar i}_{\bar l\bar\lambda} T^{k}_{j\lambda} - \frac{1}{4} T^{k}_{i\lambda} T^{\bar j}_{\bar\lambda\bar l} + \frac{1}{4} T^{\bar i}_{\bar k\bar\lambda} T^{l}_{j\lambda} \\
&\quad + \frac{1}{2} T^{\lambda}_{\bar k\bar l}(T^{\bar j}_{i\lambda} + T^{\bar i}_{\lambda j} - T^{\bar\lambda}_{ji}) + \frac{1}{2} T^{\bar\lambda}_{\bar k\bar l} T^{\lambda}_{ij},
\end{aligned}$$

where we used that $\nabla_{[Z_{\bar k}, Z_{\bar l}]} Z_i(p) = 0$, the formula in Lemma 3.1 and $R_{\bar k\bar l ij} = 0$.

Since (M^{2n}, J, g) is assumed to be nearly Kähler, we have that $T^{\bar k}_{ij} = T^{\bar i}_{jk}, T^{k}_{ij} = 0$, $\nabla T = 0, \bar\nabla T = 0$ and $R^{L}_{ijk\bar l} = 0$ from Lemma 3.2, 3.3 and 3.4. Then we obtain from

(†) at p,

$$
\begin{aligned}
(\ddagger)\quad R^L_{ij\bar{k}\bar{l}} &= \frac{1}{2} T^{\lambda}_{\bar{k}\bar{l}} (T^{\bar{j}}_{i\lambda} + T^{\bar{i}}_{\lambda j} - T^{\bar{i}}_{\lambda j}) \\
&= \frac{1}{2} T^{\lambda}_{\bar{k}\bar{l}} (T^{\bar{j}}_{i\lambda} + T^{\bar{\lambda}}_{ji} - T^{\bar{j}}_{i\lambda}) \\
&= -\frac{1}{2} T^{\lambda}_{\bar{k}\bar{l}} T^{\bar{\lambda}}_{ij} \\
&= -\frac{1}{2} B^{\lambda}_{\bar{k}\bar{l}} B^{\bar{\lambda}}_{ij},
\end{aligned}
$$

which tells the G-Kähler-likeness is equivalent to $B^{\bar{\lambda}}_{ij} B^{\lambda}_{\bar{k}\bar{l}} = 0$ for all $i, j, k, l = 1, \dots, n$.

Next, we investigate the Kähler-likeness. The following result for nearly Kähler manifolds has been proven in [**17**].

Lemma 3.6. *([17, Corollary 3.5]) Let (M, J, g) be a nearly Kähler manifold and fix a unitary frame. Then*

$$
R^L_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} + \frac{1}{4} T^{\bar{\lambda}}_{ik} T^{\lambda}_{\bar{j}\bar{l}}.
$$

We have from Lemma 3.6, we get

$$
R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = R^L_{i\bar{j}k\bar{l}} - R^L_{k\bar{j}i\bar{l}} - \frac{1}{2} T^{\bar{\lambda}}_{ik} T^{\lambda}_{\bar{j}\bar{l}}.
$$

By the Bianchi identity; $R^L_{i\bar{j}k\bar{l}} + R^L_{\bar{j}ki\bar{l}} + R^L_{ki\bar{j}\bar{l}} = 0$, we obtain from (‡),

$$
R^L_{i\bar{j}k\bar{l}} - R^L_{k\bar{j}i\bar{l}} = -R^L_{ki\bar{j}\bar{l}} = -\frac{1}{2} B^{\bar{\lambda}}_{ik} B^{\lambda}_{\bar{j}\bar{l}}.
$$

Combining these, we have

$$
R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = -\frac{1}{2} B^{\bar{\lambda}}_{ik} B^{\lambda}_{\bar{j}\bar{l}} - \frac{1}{2} T^{\bar{\lambda}}_{ik} T^{\lambda}_{\bar{j}\bar{l}} = -B^{\bar{\lambda}}_{ik} B^{\lambda}_{\bar{j}\bar{l}},
$$

which indicates the Kähler-likeness is equivalent to $B^{\bar{\lambda}}_{ik} B^{\lambda}_{\bar{j}\bar{l}} = 0$ for all $i, j, k, l = 1, \dots, n$.

Therefore, in summary, both the Kähler-likeness and the G-Kähler-likeness are equivalent to that $B^{\bar{\lambda}}_{ij} B^{\lambda}_{\bar{k}\bar{l}} = 0$ for all $i, j, k, l = 1, \dots, n$, which tells us that the Kähler-likeness is equivalent to the G-Kähler-likeness on (M^{2n}, J, g). □

*Proof of Theorem 1.4***.** Let $\Phi(X, Y, Z) = g(T(X, Y), Z)$ for any $(1,0)$-vectors X, Y and Z. Then Φ is a $(3,0)$-form on M by Lemma 3.2. Let Z_1, Z_2, Z_3 be a unitary frame and $\zeta^1, \zeta^2, \zeta^3$ be its dual frame. Suppose that $\Phi = c\zeta^1 \wedge \zeta^2 \wedge \zeta^3$. Then we have that

$$
T^{\bar{k}}_{ij} = c \cdot \operatorname{sgn} \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}.
$$

As we see in the proof of Proposition 1.3, since both the Kähler-likeness and the G-Kähler-likeness are equivalent to that $B^{\bar{\lambda}}_{ik} B^{\lambda}_{\bar{j}\bar{l}} = 0$ for $i, j, k, l = 1, \dots, n$, we have that

$$
0 = g^{k\bar{l}} B^{\bar{\lambda}}_{ik} B^{\lambda}_{\bar{j}\bar{l}} = T^{\bar{\lambda}}_{ik} T^{\lambda}_{\bar{j}\bar{k}} = |c|^2 \cdot g_{i\bar{j}}.
$$

Note that since T is parallel by Lemma 3.3, $|T|^2$ is constant, which means that $|c|^2$ does not depend on the point. In both cases, we obtain $c = 0$, which tells us that we have $T_{ij}^{\bar{k}} = 0$ for all $i, j, k = 1, \ldots, n$. Since $T'' = 0$ is equivalent to that J is integrable, we conclude that M must be Kähler. □

Remark 3.7. The proof of Theorem 1.4 uses the fact that an isotropic vector in an inner product spaces is zero, suggesting that the generalization of the main result to nearly Kähler structure of arbitrary signature might be false.

Appendix

We give a proof of Lemma 1.6 for readers' convenience.

Proof of Lemma 1.6. (1) Assume that the Chern connection ∇ is Kähler-like in the sense of Definition 1.5. By using (1Bnc), we obtain

$$R(X, \bar{Y}, Z, \bar{W}) = -R(\bar{Y}, Z, X, \bar{W}) - R(Z, X, \bar{Y}, \bar{W}) = R(Z, \bar{Y}, X, \bar{W})$$

for any $(1, 0)$-tangent vector fields X, Y, Z and W. On the other hand, assume that the almost Hermitian structure is Kähler-like in the sense of Definition 1.2. Since the Chern connection satisfies (Cplx), it suffices to show (1Bnc) for

$$(x, y, z, w) \in \{(\bar{X}, Y, Z, \bar{W}), (X, \bar{Y}, Z, \bar{W}), (X, Y, \bar{Z}, \bar{W})\},$$

where X, Y, Z and W are $(1, 0)$-tangent vector fields. For instance, in the first case, we have

$$\begin{aligned} &R(\bar{X}, Y, Z, \bar{W}) + R(Y, Z, \bar{X}, \bar{W}) + R(Z, \bar{X}, Y, \bar{W}) \\ =& - R(Y, \bar{X}, Z, \bar{W}) + R(Z, \bar{X}, Y, \bar{W}) = 0. \end{aligned}$$

(2) Assume that the Levi-Civita connection D is Kähler-like in the sense of Definition 1.5. By using (Cplx), we obtain for any $(1, 0)$-tangent vector fields X, Y, Z, and W that

$$R^L(X, Y, \bar{Z}, \bar{W}) = R^L(X, Y, J\bar{Z}, J\bar{W}) = -R^L(X, Y, \bar{Z}, \bar{W})$$

for $(1, 0)$-tangent vector fields X, Y, Z and W. Hence we get $R^L(X, Y, \bar{Z}, \bar{W}) = 0$. Furthermore, by applying $R^L(x, y, z, w) = R^L(z, w, x, y)$ we have

$$R^L(X, Y, Z, \bar{W}) = R^L(Z, \bar{W}, X, Y) = R^L(Z, \bar{W}, JX, JY) = -R^L(X, Y, Z, \bar{W}).$$

Therefore $R^L(X, Y, Z, \bar{W}) = 0$ for all $(1, 0)$-tangent vector fields X, Y, Z and W.

On the other hand, assume that the almost Hermitian structure is G-Kähler-like in the sense of Definition 1.2. For any $(1, 0)$-tangent vector fields X, Y, Z, and W, $R^L(X, Y, Z, W) = 0$ (cf. [**9**]). Since the Levi-Civita connection satisfies (1Bnc), it suffices to show $R^L(x, y, z, w) = R^L(x, y, Jz, Jw)$. Then the terms of curvatures are reduced as follows: $R^L(\bar{X}, Y, Z, \bar{W})$, $R^L(X, \bar{Y}, Z, \bar{W})$, and

$$\begin{aligned} &R^L(X, Y, Z, \bar{W}) = R^L(X, Y, \bar{Z}, \bar{W}) = R^L(\bar{X}, Y, \bar{Z}, \bar{W}) \\ =&R^L(X, \bar{Y}, \bar{Z}, \bar{W}) = R^L(\bar{X}, \bar{Y}, Z, \bar{W}) = R^L(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = 0. \end{aligned}$$

Hence in this case, the condition (Cplx) holds. □

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