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LOWER BOUNDS FOR CORNER-FREE SETS

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Abstract. A *corner* is a set of three points in $\mathbf{Z}^2$ of the form $(x, y), (x + d, y), (x, y + d)$ with $d \neq 0$. We show that for infinitely many N there is a set $A \subset [N]^2$ of size $2^{-(c+o(1))\sqrt{\log_2 N}} N^2$ not containing any corner, where $c = 2\sqrt{2\log_2 \frac{4}{3}} \approx 1.822\ldots$.

Let q, d be large positive integers. For each $x \in [q^d - 1]$, we may write $\pi(x) = (x_0, \ldots, x_{d-1}) \in \mathbf{Z}^d$ for the vector of digits of its base q expansion, thus $x = \sum_{i=0}^{d-1} x_i q^i$, with $0 \leqslant x_i < q$ for all i.

For each positive integer r, consider the set A_r of all pairs $(x, y) \in [q^d - 1]^2$ for which $\|\pi(x) - \pi(y)\|_2^2 = r$ and $\frac{q}{2} \leqslant x_i + y_i < \frac{3q}{2}$ for all i.

We claim that A_r is free of corners. Suppose that $(x, y), (x+d, y), (x, y+d) \in A_r$. Then

$$\|\pi(x) - \pi(y)\|_2^2 = \|\pi(x + d) - \pi(y)\|_2^2 = \|\pi(x) - \pi(y + d)\|_2^2 = r. \tag{1}$$

We claim that

$$\pi(x + d) + \pi(y) = \pi(x) + \pi(y + d). \tag{2}$$

To this end, we show that $(x + d)_i + y_i = x_i + (y + d)_i$ for $i = 0, 1, \ldots$ by induction on i. A single argument works for both the base case $i = 0$ and the inductive step. Suppose that, for some $j \geqslant 0$, we have the statement for $i < j$. Write $x_{\geqslant j} := \sum_{i \geqslant j} x_i q^i$, and define $(x+d)_{\geqslant j}, y_{\geqslant j}, (y+d)_{\geqslant j}$ similarly. By the inductive hypothesis and the fact that $x+(y+d) = (x+d)+y$, we see that $x_{\geqslant j}+(y+d)_{\geqslant j} = (x+d)_{\geqslant j}+y_{\geqslant j}$. Therefore $x_j + (y + d)_j = (x + d)_j + y_j (\mathrm{mod}\ q)$. However by assumption we have $\frac{q}{2} \leqslant x_j + (y + d)_j, (x + d)_j + y_j < \frac{3q}{2}$, and so $x_j + (y + d)_j = (x + d)_j + y_j$. The induction goes through.

With (2) established, let us return to (1). We now see that this statement implies that $\|a\|_2^2 = \|a + b\|_2^2 = \|a - b\|_2^2 = r$, where $a := \pi(x) - \pi(y)$ and $b := \pi(x + d) - \pi(x) = \pi(y+d)-\pi(y)$. By the parallelogram law $2\|a\|_2^2+2\|b\|_2^2 = \|a-b\|_2^2+\|a+b\|_2^2$, this immediately implies that $b = 0$. Since π is injective, it follows that $d = 0$ and so indeed A_r is corner-free.

The set of all pairs (x, y) with $\frac{q}{2} \leqslant x_i + y_i < \frac{3q}{2}$ for all i has size $(\frac{3}{4}q^2 + O(q))^d$. Therefore by the pigeonhole principle there is some r such that $\#A_r \geqslant (dq^2)^{-1}(\frac{3}{4}q^2 + O(q))^d$.

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Now for a given d set $q := \lfloor (2/\sqrt{3})^d \rfloor$ and $N := q^d$. Then $A_r \subset [N] \times [N]$, A_r is free of corners, and

$$\# A_r \geqslant N^2 (dq^2)^{-1} (\frac{3}{4} + O(\frac{1}{q}))^d.$$

Writing $o(1)$ for a quantity tending to 0 as $N \to \infty$, we note that $q = (\frac{2}{\sqrt{3}} + o(1))^d$ and that $d = (1 + o(1))\sqrt{\frac{\log_2 N}{\log_2 (2/\sqrt{3})}}$. A short calculation then confirms that

$$\# A_r \geqslant N^2 2^{-(c + o(1))\sqrt{\log_2 N}},$$

where $c = 2\sqrt{2 \log_2 \frac{4}{3}} \approx 1.822\dots$.

Remark. The construction came about by a careful study of the recent preprint of Linial and Shraibman [**1**], where they used ideas from communication complexity to obtain a bound with $c = 2\sqrt{\log_2 e} \approx 2.402\dots$, improving on the previously best known bound with $c = 2\sqrt{2} \approx 2.828\dots$ which comes from Behrend's construction. By bypassing the language of communication complexity one may simplify the construction, in particular avoiding the use of entropy methods. This yields a superior bound.

References

[1] N. Linial and A. Shraibman, *Larger corner-free sets from better NOF exactly-N protocols,* preprint, arxiv:2102.00421.

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