NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 50 (2020), 11-19

FIBERED BOUNDARIES AND CALORONS

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(Received 8 November, 2019)

Abstract. We use the index formula for fibered boundaries to compute the L^2-index of the Dirac operator twisted by an Anti-Self-Dual instanton defined on $X \times S^1$, where X is a complete asymptotically conical three-manifold. As a particular case of this calculation, we get another derivation of the index formula for the Dirac operator twisted by a caloron on $\mathbb{R}^3 \times S^1$.

Introduction

Given a complete orientable, Riemannian four-manifold $\mathcal{M}$ and a complex vector bundle $\mathcal{E} \to \mathcal{M}$, a unitary connection $\mathfrak{A}$ on $\mathcal{E}$ is called *instanton* if its curvature $\mathfrak{F}_{\mathfrak{A}}$ is square integrable and Anti-Self-Dual with respect to the chosen orientation. Understanding the moduli space of instantons on $\mathcal{M}$ is an important problem in gauge theory. One powerful technique to study this moduli space is called *Nahm Transform.* This is a map, defined using index theory of Dirac operators, that relates instantons on $\mathcal{M}$ to solutions of a system of algebraic equations defined on a different space $\hat{\mathcal{M}}$. A recent example of Nahm transform is the series of papers [**4**], [**5**] and [**6**], where it is proved that the moduli space of instantons on Multi-Taub-NUT spaces is isometric to the moduli space of *Bow Representations* [**3**].

One important intermediate step in the Nahm transform is the computation of the index of the Dirac operator twisted by the instanton. This index has been computed, depending on the base space $\mathcal{M}$, by several different methods. In this paper we illustrate how the index formula for twisted Dirac operators on manifolds with fibered boundaries [**8**] can be used on certain open four-manifolds which are circle fibrations near infinity.

More concretely, we consider instantons on $X \times S^1$, where X is an almost conical three-manifold (see definition 4). These spaces are generalizations of $\mathbb{R}^3 \times S^1$. Instantons over $\mathbb{R}^3 \times S^1$ are called *calorons* in the physics literature. The original computation of the index for calorons was given in [**9**]. Here, the authors conjectured a relation between this index and the adiabatic limit of the η-invariant of the Dirac operator on the boundary[1]. The index formula of [**8**] proves that, since the adiabatic limit of η is related to the $\hat{\eta}$-form of [**2**], their conjectured relation was correct.

The motivation to study calorons on $X \times S^1$ is that, at least for $SU(2)$-calorons, there are explicit nontrivial examples. These examples come from [**10**], where monopoles on X are constructed. The asymptotic conditions that we impose on calorons are the ones that most known examples satisfy.

2010 *Mathematics Subject Classification* 58J20, 53C07.
Key words and phrases: Index theory, Eta form, Caloron, monopole.

[1]This relation appears in a preprint version of [**9**] but not in the published version.

Our main result (theorem 11) is

Theorem. *Let $\mathfrak{A}$ be a generic unitary AC-caloron connection with curvature $\mathfrak{F}$ (see definition 7) then the index of the twisted Dirac operator $\mathcal{D}^+_{\mathfrak{A}}$ is given by*

$$\mathrm{Ind}_{L^2}(\mathcal{D}^+) = \int_{X\times S^1} \big(\frac{\mathrm{Rank}(\mathcal{E})}{192\pi^2}\mathrm{Tr}\, R_g \wedge R_g - \frac{1}{8\pi^2}\mathrm{Tr}\,\mathfrak{F}\wedge\mathfrak{F}\big) - \sum_{j=1}^{n}(1/2 - \{\lambda_j\})m_j. \quad (1)$$

This paper starts with a review of the material required to state the index formula of [**8**]. In section 2, we define the instantons on $X \times S^1$, called *generic AC-calorons*, that we consider and calculate the index of the corresponding twisted Dirac operator.

1. Index Theory on Spaces with Fibered Boundaries

We start reviewing the index theorem in [**8**]. This statement writes the index of a twisted Dirac operator as a sum of two terms. The first one, called the *bulk term*, is the classical Atiyah-Singer integrand and the other one is an integral over the boundary at infinity of an expression involving the $\hat{\eta}$-form defined by Bismut-Cheeger [**2**].

1.1. The Bismut-Cheeger $\hat{\eta}$-form. Let $\pi : M \to B$ be a locally trivial fibration of closed spin manifolds with fibers isomorphic to a manifold Z. We assume there is a connection on the fibration that induces a splitting $TM = T_H M \oplus TM/B$ into horizontal and vertical tangent vectors, such that $\pi^* TB$ can be identified with $T_H M$. Let $g^M = \pi^* g^B \oplus g^{M/B}$ be a Riemannian submersion metric, where g^B is a metric on TB pulled back to $T_H M$, and $g^{M/B}$ denotes a metric on the vertical fibers. We use $\{f_\alpha\}$, $\{f^\alpha\}$ to denote a frame on TB and T^*B respectively. Also, $\{e_j\}$, $\{e^j\}$ denote frames on TM/B and T^*M/B respectively.

Let $E \to M$ be a complex vector bundle with unitary connection ∇^E and curvature F^E. The bundle E induces an infinite rank bundle $\pi_* E \to B$ with fibers given by $\Gamma(M_x, E_x)$, where M_x, E_x denote the fibers over $x \in B$. The connection ∇^E induces a connection on $\pi_* E$ denoted by $\nabla^{\pi_* E}$ [**1**, Chap 10].

Let $(\mathcal{S}^{M/B}, \nabla^{M/B})$ be the vertical spinor bundle together with its induced spin connection coming from the metric $g^{M/B}$. We denote by c^j the Clifford product $c(e^j)$ in $\mathcal{S}^{M/B}$. We use $\nabla^{\mathcal{S}^{M/B}\otimes E} = \nabla^{M/B} \otimes 1 + 1 \otimes \nabla^E$ and the Clifford module structure on $\mathcal{S}^{M/B} \otimes E$, with respect to the Clifford algebra of TM/B, to construct a family of vertical Dirac operators denoted by $D^{M/B} = c^{M/B} \circ \nabla^{\mathcal{S}^{M/B}\otimes E}$. Here, $c = c^{M/B}$ denotes the Clifford product by elements of T^*M/B.

Definition 1. [**1**, prop. 10.15] *Let u be a positive parameter, the Bismut superconnection, acting on $\Gamma(M, \mathcal{S}^{M/B} \otimes E) = \Gamma(B, \pi_*(\mathcal{S}^{M/B} \otimes E))$, is defined by*

$$\mathfrak{A}_u = \nabla^{\pi_*(\mathcal{S}^{M/B}\otimes E)} + \sqrt{u} D^{M/B} - \frac{c(T)}{4\sqrt{u}}, \quad (2)$$

where T is the torsion form of the fibration $M \to B$.

Definition 2. *The Bismut-Cheeger Eta form of the vertical family of Dirac operators $D^{M/B}$ is defined according to the dimension of the base manifold B.*

- *If* $\dim(B)$ *is even then*
$$\hat{\eta}(D^{M/B}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}^{\mathrm{ev}}\big((D^{M/B} + \frac{c(T)}{4u})e^{-\mathfrak{A}_u^2}\big)\frac{du}{2\sqrt{u}}, \tag{3}$$
where $\mathrm{Tr}^{\mathrm{ev}}$ *denotes the operator trace on the even form part of* $\hat{\eta}$.
- *If* $\dim(B)$ *is odd then*
$$\hat{\eta}(D^{M/B}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}^{s}\big((D^{M/B} + \frac{c(T)}{4u})e^{-\mathfrak{A}_u^2}\big)\frac{du}{2\sqrt{u}}, \tag{4}$$
where Tr^s *denotes the supertrace.*

1.2. The index formula. Consider a Riemannian manifold $(\mathcal{M}, g^{\mathcal{M}})$ such that its boundary $M = \partial\mathcal{M}$ is the total space of a fibration $\pi : M \to B$ like the one considered in section 1.1.

We assume that on a tubular neighborhood of the boundary $(a, \infty)_y \times M$, the metric takes the form
$$g^{\mathcal{M}} = dy^2 + \pi^* g^B + e^{-2y} g^{M/B}. \tag{5}$$

In the terminology of [**8**], this is an exact d-metric with boundary defining function $x = e^{-y}$.

Let
$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} \tag{6}$$
be a Dirac type operator (for example a twisted Dirac operator) on $\mathcal{S} \otimes \mathcal{E} \to \mathcal{M}$ (here $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ denotes the spin bundle), such that its boundary family $D^{M/B}$, acting on $\mathcal{S}^{M/B} \otimes (\mathcal{E}|_M) = \mathcal{S}^{M/B} \otimes E$, satisfies the technical assumption
$$\mathrm{Spec}(\mathrm{D_b^{M/B}}) \cap (-\delta, \delta) = \emptyset, \tag{7}$$
for some $\delta > 0$ and for every $b \in B$.

We can now state the index theorem that we will use in this work.

Theorem 3. [**8**] *If $D^{M/B}$ satisfies assumption (7) then*
$$\mathrm{Ind}_{L^2}(\mathcal{D}^+) = \int_{\mathcal{M}} \hat{A}(\mathcal{M}, g^{\mathcal{M}}) \wedge \mathrm{Ch}(\mathcal{E}) - \frac{1}{2\pi i}\int_{\partial\mathcal{M}} \hat{A}(B, g^B) \wedge \hat{\eta}(D^{M/B}), \tag{8}$$
where $\hat{A}$ denotes the A-hat genus of $\mathcal{M}$. The $\hat{\eta}$-form is computed with respect to the submersion metric $\pi^* g^B + g^{M/B}$.

The $2\pi i$ factor does not appear explicitly in the original formula due to the use of different normalizations. See [**7**].

1.3. An important identity. Let z be an auxiliary variable such that $z^2 = 0$. In this paper, we will only encounter manifolds whose fibered boundaries are trivial fibrations $M = B \times Z \to B$, where Z is a closed manifold with vanishing scalar curvature. The following identity will play a crucial role in the calculations of section 2.

$$\mathfrak{A}_u^2 - z\sqrt{u}D^{M/B} = -u\sum_j \left(\nabla_{e_j} + \frac{zc^j}{2\sqrt{u}}\right)^2 + \frac{u}{2}c_i c_j F^E(e_i, e_j) + \sqrt{u}c^j F^E(f_\alpha, e_j)f^\alpha + \frac{1}{2}F^E(f_\alpha, f_\beta)f^\alpha \wedge f^\beta. \quad (9)$$

Notice that the last three terms of the first line are forms on B. This is a particular case of [**2**, (4.68)-(4.70)].

2. Almost Conical Calorons

Here we review results of [**10**] that prove existence of solutions to Bogomolny equations over almost conical three manifolds. These existence results imply that the corresponding space of calorons is nonempty.

2.1. Monopoles. We begin reviewing some properties of the metrics on almost conical three-manifolds.

Definition 4. [**10**] *A complete, orientable three-manifold X is said to be almost-conical or AC with one end if it admits a metric g_{AC} such that there is a compact set K in X for which $X \setminus K$ is isometric to a cylinder $(r_0, \infty) \times \Sigma$. Here, Σ is a closed Riemann surface and $r_0 > 0$. The metric on the cylinder is $g = dr^2 + r^2 g_\Sigma$, where g_Σ is a metric on Σ.*

We only consider one cylindrical end to simplify notation. The case of multiple cylindrical ends can be analyzed similarly. The flat metric on $\mathbb{R}^3$ in polar coordinates is an example of an AC-manifold with $\Sigma = S^2$. Notice that AC-metrics have cubic volume growth. From now on, (X, g_{AC}) denotes an AC-three manifold with one end.

Definition 5. [**10**] *Let $E \to X$ be a unitary bundle over an AC-three manifold X. A monopole on E is a pair (A, Φ) of a unitary connection on E and a section $\Phi \in \Gamma(X, \mathrm{End}(E))$ such that $F_A = -\star_3 d_A\Phi$. This is called the Bogomolny equation. Also, F_A and $d_A\Phi$ should be L^2.*

The following result establishes the existence of $SU(2)-$monopoles on X satisfying explicit asymptotic conditions.

Theorem 6. [**10**] *There are $SU(2)$-monopoles (A, Φ) on X, such that*

$$|\Phi| = \lambda + \frac{m}{r} + \mathcal{O}(r^{-2}), \quad (10)$$

where λ is a real number and m is an integer.

Actually, the results of [**10**] are much stronger and include a characterization of the moduli space of $SU(2)$-monopoles. Here we only need this basic existence statement.

Note: In this paper, whenever we say that a function satisfies $f = \mathcal{O}(r^{-2})$, it is also assumed that the derivatives of f also satisfy this bound.

2.2. Generic AC-calorons. Given a monopole (A,Φ) on X, we can construct an ASD instanton on $X \times S^1$ by $\mathfrak{A} = A + \Phi\, d\tau$, where τ parametrizes S^1 and is 2π-periodic. Instantons on $X\times S^1$ are called almost conical calorons (AC-calorons). The theorem above also proves that the space of $SU(2)$ AC-calorons is not empty.

In [**9**], the authors study more general $U(n)$-calorons on $\mathbb{R}^3 \times S^1$ imposing the asymptotic conditions

$$\Phi = -i\mathrm{Diag}(\lambda_j + \frac{m_j}{r}) + \mathcal{O}(r^{-2}), \tag{11}$$

where $\lambda_j \in \mathbb{R}$ and $m_j \in \mathbb{Z}$. This implies that the bundle $\mathcal{E}$ decomposes, near infinity, as a sum of line bundles with First Chern classes m_j, over the sphere at infinity $\Sigma = S^2_\infty$, pulled back to $S^2_\infty \times S^1$. It is a consequence of the results in [**4**], that generic calorons, as well as generic instantons over any ALF-space with Riemannian curvature decaying faster than quadratically, have the asymptotic form (11).

Here we study $U(n)$-instantons on $X \times S^1$ with similar asymptotics.

Definition 7. *Given a complex bundle $\mathcal{E} \to X \times S^1$, a unitary connection $\mathfrak{A}$ is a generic AC-caloron if its curvature $\mathfrak{F}_{\mathfrak{A}}$ is ASD and square integrable. Furthermore, we assume that there is a frame of $\mathcal{E}$, defined outside a compact set $K \subset X\times S^1$, such that $\mathfrak{A} = A + \Phi\, d\tau + \mathcal{O}(r^{-2})$, where (A,Φ) are $\tau-$independent, satisfy Bogomolny equation and Φ has the same asymptotics as* (11)*. We assume that the λ_j are real numbers, pairwise distinct and not integer valued.*

This last restriction is called *Maximal Symmetry Breaking.* Notice that the $m_j \in \mathbb{Z}$ are now the Chern numbers of line bundles over a more general Riemann surface Σ at infinity. We assume $X \times S^1$ is a spin manifold with a fixed spin structure. The following lemmas prove several important properties of the twisted Dirac operator $\mathcal{D}_{\mathfrak{A}}$.

We start by analyzing the decay rate of the Riemannian curvature tensor of the metric.

Lemma 8. *The curvature of the metric $g = g_{\mathrm{AC}} + d\tau^2 = d^2r + r^2 g_\Sigma + d^2\tau$, on $X \times S^1$, decays quadratically.*

Proof. Let $\{f^2, f^3\}$ an orthonormal coframe on Σ with respect to g_Σ. Notice that $df^2 = \beta f^2 \wedge f^3$ and $df^3 = \gamma f^2 \wedge f^3$, where β and γ are functions that only depend on the coordinates on Σ.

Define an orthonormal coframe of $X \times S^1$ by $\theta^1 = dr$, $\theta^2 = rf^2$, $\theta^3 = rf^3$ and $\theta^4 = d\tau$.

It follows that $df^2 = r^{-2}\beta\theta^2 \wedge \theta^3 = \mathcal{O}(r^{-2})$ and $df^3 = r^{-2}\gamma\theta^2 \wedge \theta^3 = \mathcal{O}(r^{-2})$ in the metric g_{AC}. Notice that $\beta, \gamma = \mathcal{O}(1)$ since they are independent of r.

Let θ be the vector of one-forms with components θ^j. From Cartan's structural equations $d\theta = -\omega \wedge \theta$ it follows that the matrix of one forms ω equals

$$\omega = \begin{pmatrix} 0 & -\theta^2/r & -\theta^3/r & 0 \\ \theta^2/r & 0 & -\beta f^2 - \gamma f^3 & 0 \\ \theta^3/r & \beta f^2 + \gamma f^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Every component of ω is $\mathcal{O}(r^{-1})$, hence products of components lie in $\mathcal{O}(r^{-2})$. Also, the derivative of each component is in $\mathcal{O}(r^{-2})$. It follows that the curvature $d\omega + \omega \wedge \omega$ is $\mathcal{O}(r^{-2})$. □

An important consequence of this lemma is that the scalar curvature of g_{AC} also decays quadratically.

Now we establish the decay rate of the curvature of a generic AC-caloron.

Lemma 9. *Let $\mathfrak{A} = A + \Phi\, d\tau + \mathcal{O}(r^{-2})$ be a generic AC-caloron defined on a bundle $\mathcal{E} \to X \times S^1$ then its curvature $\mathfrak{F}$ decays quadratically.*

Proof. We have that

$$\mathfrak{F} = F_A + d_A\Phi \wedge d\tau + \mathcal{O}(r^{-2}). \tag{12}$$

From (11) it follows that $d_A\Phi = i\mathrm{diag}(m_j/r^2) + \mathcal{O}(r^{-2}) = \mathcal{O}(r^{-2})$. The Bogomolny equation $F_A = - \star_3 d_A\Phi$ implies that $F_A = \mathcal{O}(r^{-2})$. Since $d\tau = \mathcal{O}(1)$, the result follows. □

Using these decay rates, we can establish important properties of Dirac operators twisted by generic AC-calorons.

Lemma 10. *The twisted Dirac operator $\mathcal{D}_{\mathfrak{A}}$ is Fredholm and the L^2-solutions of $\mathcal{D}_{\mathfrak{A}}\psi = 0$ decay exponentially.*

Proof. The lemma is a particular case of [**4**, Lemma 23, Prop. 24].

Denote by s the scalar curvature of g_{AC}. The Lichnerowicz formula implies that for $h \in L^2(\mathcal{S} \otimes \mathcal{E})$ one has

$$\|\mathcal{D}_{\mathfrak{A}} h\|^2_{L^2} = \|\nabla^{\mathfrak{A}} h\|^2_{L^2} + \langle h, (\frac{s}{4} + c(\mathfrak{F}))h\rangle, \tag{13}$$

where $c(\mathfrak{F})$ denotes Clifford product by the curvature $\mathfrak{F}$. The condition (11) and the quadratic decay of s and $\mathfrak{F}$ imply that for a compact set K large enough there is $\alpha > 0$ such that

$$\|\nabla^{\mathfrak{A}} h\|^2_{L^2} + \langle h, (\frac{s}{4} + c(\mathfrak{F}))h\rangle > \alpha^2\|h\|^2_{L^2},$$

for every h with compact support such that supp(h) $\subset K^c$. This implies that $\mathcal{D}_{\mathfrak{A}}$ is Fredholm.

Let $\psi \in L^2(\mathcal{S} \otimes \mathcal{E})$ solve $\mathcal{D}_{\mathfrak{A}}\psi = 0$. Consider a compactly supported function η_n equal to e^{br} for $r \leq n$. The constant $b > 0$ will be determined later. Again, Lichnerowicz formula gives for some $\epsilon > 0$ (this ϵ should be smaller than all the $|\lambda_j|$ from condition (11). Remember that the λ_j are not integer valued so they can't equal zero.)

$$\begin{aligned} 0 = \|\mathcal{D}_{\mathfrak{A}}(\eta_n\psi)\|^2_{L^2} - \|c(d\eta_n)\psi\|^2_{L^2} &\geq \|\nabla^{\mathfrak{A}}(\eta_n\psi)\|^2_{L^2} - b^2\|\eta_n\psi\|^2_{L^2} - C_1\|r^{-1}\eta_n\psi\|^2_{L^2} \\ &\geq (\epsilon^2 - b^2)\|\eta_n\psi\|^2_{L^2} - C_2\|r^{-1}\eta_n\psi\|^2_{L^2}. \end{aligned}$$

We used (11) in the last inequality. Taking $0 < b < \epsilon$ implies that $\eta_n\psi \in L^2(\mathcal{S}\otimes\mathcal{E})$. Dominated convergence gives $e^{br}\psi \in L^2(\mathcal{S} \otimes \mathcal{E})$. The L^∞ bound on $e^{br}\psi$ follows from Moser iteration [**4**, Prop. 2]. □

3. The Index Computation

This section contains the proof of the main result of this paper: an index formula for Dirac operators on $X \times S^1$ twisted by generic AC-calorons.

Theorem 11. *The index of $\mathcal{D}^+_{\mathfrak{A}}$ is given by*

$$\mathrm{Ind}_{L^2}(\mathcal{D}^+) = \int_{X\times S^1} \big(\frac{\mathrm{Rank}(\mathcal{E})}{192\pi^2}\mathrm{Tr}\, R_g \wedge R_g - \frac{1}{8\pi^2}\mathrm{Tr}\,\mathfrak{F}\wedge\mathfrak{F}\big) - \sum_{j=1}^{n}(1/2-\{\lambda_j\})m_j, \quad (14)$$

where R_g is the curvature of the Riemannian metric g.

Proof. We start with a conformal change to g_{AC}. Take a function u such that $e^{2u} = r^{-2}$ for r large on the cylindrical end $X \times S^1 \setminus K$, and e^{2u} equal to 1 on K. Changing variables $r = e^y$, on the cylindrical end, the new metric $g' = e^{2u}g$ has the form

$$g' = d^2y + g_\Sigma + e^{-2y}d^2\tau. \quad (15)$$

This is an exact d-metric in the terminology of [**8**]. The boundary fibration is a trivial circle fibration $\Sigma \times S^1 \to \Sigma$. The exponential decay of the solutions to $\mathcal{D}_{\mathfrak{A}}\psi = 0$ and the transformation formula for Dirac operators under conformal changes of the metric [**4**, Prop. 30], imply that the index remains the same if we use the new metric g'. Notice also that the fiberwise Dirac operators D^{S^1} satisfy condition (7) since the λ_j are not integers and they are also pairwise distinct. Actually, setting $c(d\tau) = c(e) = -i$ we get

$$D^{S^1} = \oplus_{j=1}^{n}(-i)(\partial_\tau - i\lambda_j), \quad (16)$$

which is invertible if $\lambda_j \notin \mathbb{Z}$ for all j. Since the boundary fibration is trivial, there is no torsion form T.

The $\hat{\eta}$-form in this case is

$$\hat{\eta}(D^{S^1}) = \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{Tr}^{\mathrm{ev}}\big(D^{S^1}e^{-\mathfrak{A}_u^2}\big)\frac{du}{2\sqrt{u}}, \quad (17)$$

The bundle $\mathcal{E}$ decomposes at $y = \infty$ as a direct sum of n line bundles, originally defined over Σ and pulled back to $\Sigma \times S^1$, where the m_j are the first Chern classes of each bundle. This implies that the identity (9) becomes in this case

$$-u\big(\nabla_{\partial_\tau} - \frac{zi}{2\sqrt{u}}\big)^2 + F^E(f_2, f_3)f^2 \wedge f^3 = \mathfrak{A}_u^2 - z\sqrt{u}D^{S^1}. \quad (18)$$

In what follows, we denote $F^E(f_2, f_3)f^2 \wedge f^3$ by F.

Let $\mathrm{Tr}^z(a + bz) = \mathrm{Tr}\, b$, where a, b do not contain z, then exponentiating both sides of (18) we get [**2**, (4.73)]

$$\mathrm{Tr}^{\mathrm{ev}} D^{S^1}e^{-\mathfrak{A}_u^2} = \frac{1}{\sqrt{u}}\mathrm{Tr}^z \exp\{u(\nabla_{\partial_\tau} - \frac{zi}{2\sqrt{u}})^2 - F\}. \quad (19)$$

Let $\Lambda = \mathrm{Diag}(\lambda_j)$ be the asymptotic matrix of $\mathfrak{A}(\partial_\tau)$ over $\Sigma_\infty \times S^1$. We expand in Fourier series with respect to τ. Then the right hand side of (19) equals

$$\frac{1}{\sqrt{u}}\mathrm{Tr}^z\sum_{k\in\mathbb{Z}}\exp\{-u(k - \Lambda - \frac{z}{2\sqrt{u}})^2 - F\}. \quad (20)$$

This in turn equals

$$\mathrm{Tr}^E \sum_{k\in\mathbb{Z}} e^{-F}(k-\Lambda)e^{-u(k-\Lambda)^2}. \tag{21}$$

The Poisson summation formula transforms (21) into

$$-\mathrm{Tr}^E e^{-F} \sum_{p\geq 1} 2p(\sin 2\pi p\Lambda)\frac{\pi^{3/2}}{u^{3/2}}e^{-\frac{\pi^2p^2}{u}}. \tag{22}$$

Therefore,

$$\begin{aligned}\hat{\eta}(D^{S^1}) &= \frac{-1}{\sqrt{\pi}}\int_0^\infty \mathrm{Tr}^E e^{-F}\sum_{p\geq 1}\pi p(\sin 2\pi p\Lambda)e^{-\frac{\pi^2p^2}{u}}\frac{du}{u^2} \\ &= -\mathrm{Tr}^E e^{-F}\sum_{p\geq 1}\frac{\sin 2\pi p\Lambda}{\pi p} \\ &= -\mathrm{Tr}^E e^{-F}(1/2-\{\Lambda\}).\end{aligned} \tag{23}$$

Here $\{\Lambda\}$ denotes the diagonal matrix whose entries are the fractional parts of the diagonal entries of Λ. For the last line above we used the Fourier series expansion of Bernoulli polynomials.

The index formula gives

$$\mathrm{Ind}_{L^2}(\mathcal{D}^+) = \int_{X\times S^1}\big(\frac{\mathrm{Rank}(\mathcal{E})}{192\pi^2}\mathrm{Tr}\, R_g\wedge R_g - \frac{1}{8\pi^2}\mathrm{Tr}\,\mathfrak{F}\wedge\mathfrak{F}\big) + \frac{1}{2\pi i}\int_\Sigma \mathrm{Tr}^E(1/2-\{\Lambda\})e^{-F}. \tag{24}$$

Computing the last boundary integral we obtain

$$\mathrm{Ind}_{L^2}(\mathcal{D}^+) = \int_{X\times S^1}\big(\frac{\mathrm{Rank}(\mathcal{E})}{192\pi^2}\mathrm{Tr}\, R_g\wedge R_g - \frac{1}{8\pi^2}\mathrm{Tr}\,\mathfrak{F}\wedge\mathfrak{F}\big) - \sum_{j=1}^n (1/2-\{\lambda_j\})m_j. \tag{25}$$

□

As a simple application of theorem 11, notice that for $X=\mathbb{R}^3$, the first summand above vanishes and we recover, with equivalent notation, the index theorem of [**9**].

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