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GENERALIZATION OF A REAL-ANALYSIS RESULT TO A CLASS OF TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper, we generalize an elementary real-analysis result to a class of topological vector spaces. We also give an example of a topological vector space to which the result cannot be generalized.

1. Introduction

This paper draws its inspiration from the following result, which appears to be a popular real-analysis exam problem (see [**3**], for example):

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $\lim_{n\to\infty}(2x_{n+1} - x_n) = x$ for some $x \in \mathbb{R}$, then $\lim_{n\to\infty} x_n = x$.

An expedient proof can be given using the Stolz-Cesàro Theorem as follows:

Proof. Define sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ by

$$\forall n \in \mathbb{N}: \qquad a_n \stackrel{\text{df}}{=} 2^n x_n \qquad \text{and} \qquad b_n \stackrel{\text{df}}{=} 2^n.$$

Then $(b_n)_{n\in\mathbb{N}}$ is strictly increasing and diverges to ∞, and as

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} &= \lim_{n\to\infty} \frac{2^{n+1}x_{n+1} - 2^n x_n}{2^{n+1} - 2^n} \\ &= \lim_{n\to\infty} \frac{2^{n+1}x_{n+1} - 2^n x_n}{2^n} \\ &= \lim_{n\to\infty} (2x_{n+1} - x_n) \\ &= x,\end{aligned}$$

the Stolz-Cesàro Theorem immediately tells us that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a_n}{b_n} = x$. □

A natural question to ask is: Is this result still valid if $\mathbb{R}$ is replaced by another topological vector space? The answer happens to be affirmative for a wide class of topological vector spaces that includes all the locally convex ones.

We will also exhibit a topological vector space for which the result is not valid, which indicates that it is rather badly behaved.

In this paper, we adopt the following conventions:

- $\mathbb{N}$ denotes the set of all positive integers, and for each $n \in \mathbb{N}$, let $[n] \stackrel{\text{df}}{=} \mathbb{N}_{\leq n}$.
- All vector spaces are over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

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2. Good Topological Vector Spaces

Recall that a topological vector space is an ordered pair (V,τ), where:

- V is a vector space, and
- τ is a topology on V, under which vector addition and scalar multiplication are continuous operations.

Definition 2.1. Let (V,τ) be a topological vector space, and $(x_\lambda)_{\lambda\in\Lambda}$ a net in V. Then $x \in V$ is called a *τ-limit* for $(x_\lambda)_{\lambda\in\Lambda}$ — which we write as $(x_\lambda)_{\lambda\in\Lambda} \xrightarrow{\tau} x$ — if and only if for each τ-neighborhood U of x, there is a $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \in \Lambda_{\geq \lambda_0}$.

Remark 2.2. We do not assume that τ is a Hausdorff topology on V.

Definition 2.3. A topological vector space (V,τ) is said to be *good* if and only if any sequence $(x_n)_{n\in\mathbb{N}}$ in V has a τ-limit whenever $(2x_{n+1}-x_n)_{n\in\mathbb{N}}$ has a τ-limit.

A topological vector space that is not good is said to be *bad*.

Proposition 2.4. *Let (V,τ) be a topological vector space, and $(x_n)_{n\in\mathbb{N}}$ a sequence in V such that $(2x_{n+1}-x_n)_{n\in\mathbb{N}} \xrightarrow{\tau} x$ for some $x \in V$. Then either*

- *$(x_n)_{n\in\mathbb{N}} \xrightarrow{\tau} x$ also, or*
- *$(x_n)_{n\in\mathbb{N}}$ has no τ-limit.*

Proof. If $(x_n)_{n\in\mathbb{N}}$ has no τ-limit, then we are done.

Next, suppose that $(x_n)_{n\in\mathbb{N}} \xrightarrow{\tau} y$ for some $y \in V$. Then

$$(2x_{n+1}-x_n)_{n\in\mathbb{N}} \xrightarrow{\tau} 2y - y = y,$$

so y is a τ-limit for $(2x_{n+1}-x_n)_{n\in\mathbb{N}}$ in addition to x. It follows that

$$(0_V)_{n\in\mathbb{N}} = ((2x_{n+1}-x_n) - (2x_{n+1}-x_n))_{n\in\mathbb{N}} \xrightarrow{\tau} x - y,$$

which yields

$$(y)_{n\in\mathbb{N}} = (0_V + y)_{n\in\mathbb{N}} \xrightarrow{\tau} (x-y)+y = x.$$

Therefore, any τ-neighborhood of x also contains y, giving us $(x_n)_{n\in\mathbb{N}} \xrightarrow{\tau} x$. □

Proposition 2.4 tells us: To prove that a topological vector space (V,τ) is good, it suffices to prove that for each sequence $(x_n)_{n\in\mathbb{N}}$ in V, if $(2x_{n+1}-x_n)_{n\in\mathbb{N}} \xrightarrow{\tau} x$ for some $x \in V$, then $(x_n)_{n\in\mathbb{N}} \xrightarrow{\tau} x$ also.

Definition 2.5. Let $p \in (0,1]$. A *p-homogeneous seminorm* on a vector space V is then a function $\sigma : V \to \mathbb{R}_{\geq 0}$ with the following properties:

(1) **The Triangle Inequality:** $\sigma(x+y) \leq \sigma(x)+\sigma(y)$ for all $x,y \in V$.
(2) **p-Homogeneity:** $\sigma(kx) = |k|^p \sigma(x)$ for all $k \in \mathbb{K}$ and $x \in V$.

Remark 2.6. • By letting $k=0$ and $x=0_V$ in (2), we find that $\sigma(0_V)=0$.

- A 1-homogeneous seminorm is the same as a seminorm in the ordinary sense.
- No extra generality is gained by postulating that $\sigma(kx) \leq |k|^p \sigma(x)$ for all $k \in \mathbb{K}$ and $x \in V$. If $k \in \mathbb{K} \setminus \{0\}$, then replacing k by $\dfrac{1}{k}$ gives us the reverse inequality, which leads to equality; if $k=0$, then equality automatically holds.

- We do not consider $p \in (2, \infty)$ because
$$\begin{aligned} \forall x \in V: \quad 2^p \sigma(x) &= \sigma(2x) && \text{(By } p\text{-homogeneity.)} \\ &= \sigma(x + x) \\ &\leq 2\sigma(x), && \text{(By the Triangle Inequality.)} \end{aligned}$$
so if σ is non-trivial, then $2^p \leq 2$, which implies that $p \in (0, 1]$ if $p \in \mathbb{R}_{>0}$.

Let V be a vector space, and $\mathcal{S}$ a collection of p-homogeneous seminorms on V where $p \in (0, 1]$ may not be fixed. Define a function $\mathcal{U} : V \times \mathcal{S} \times \mathbb{R}_{>0} \to \mathcal{P}(V)$ by
$$\forall x \in V, \ \forall \sigma \in \mathcal{S}, \ \forall \epsilon \in \mathbb{R}_{>0}: \quad \mathcal{U}_{x,\sigma,\epsilon} \stackrel{\text{df}}{=} \{y \in V \mid \sigma(y - x) < \epsilon\}.$$
Then let $\tau_{\mathcal{S}}$ denote the topology on V that is generated by the sub-base
$$\{\mathcal{U}_{x,\sigma,\epsilon} \in \mathcal{P}(V) \mid (x, \sigma, \epsilon) \in V \times \mathcal{S} \times \mathbb{R}_{>0}\}.$$

Proposition 2.7. *The following statements about $\tau_{\mathcal{S}}$ hold:*
(1) $\tau_{\mathcal{S}}$ is a vector-space topology on V.
(2) Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in V. Then for each $x \in V$, we have
$$(x_\lambda)_{\lambda \in \Lambda} \xrightarrow{\tau_{\mathcal{S}}} x \qquad \Longleftrightarrow \qquad \lim_{\lambda \in \Lambda} \sigma(x_\lambda - x) = 0 \text{ for all } \sigma \in \mathcal{S}.$$

Proof. One only has to imitate the proof in the case of locally convex topological vector spaces that the initial topology generated by a collection of seminorms is a vector-space topology. We refer the reader to Chapter 1 of [**2**] for details. □

Proposition 2.8. *$(V, \tau_{\mathcal{S}})$ is a good topological vector space.*

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in V. Suppose that $(2x_{n+1} - x_n)_{n \in \mathbb{N}} \xrightarrow{\tau_{\mathcal{S}}} x$ for some $x \in V$. Then without loss of generality, we may assume that $x = 0_V$. To see why, define a new sequence $(y_n)_{n \in \mathbb{N}}$ in V by $y_n \stackrel{\text{df}}{=} x_n - x$ for all $n \in \mathbb{N}$, so that
$$\begin{aligned} \forall n \in \mathbb{N}: \quad 2y_{n+1} - y_n &= 2(x_{n+1} - x) - (x_n - x) \\ &= 2x_{n+1} - 2x - x_n + x \\ &= (2x_{n+1} - x_n) - x. \end{aligned}$$
Hence,
$$(2y_{n+1} - y_n)_{n \in \mathbb{N}} = ((2x_{n+1} - x_n) - x)_{n \in \mathbb{N}} \xrightarrow{\tau_{\mathcal{S}}} x - x = 0_V,$$
so if we can prove that $(y_n)_{n \in \mathbb{N}} \xrightarrow{\tau_{\mathcal{S}}} 0_V$, then $(x_n)_{n \in \mathbb{N}} \xrightarrow{\tau_{\mathcal{S}}} x$ as desired.

Let $\sigma \in \mathcal{S}$ and $\epsilon > 0$, and suppose that σ is p-homogeneous for some $p \in (0, 1]$. Then by (2) of Proposition 2.7, there is an $N \in \mathbb{N}$ such that
$$\forall n \in \mathbb{N}_{\geq N}: \quad \sigma(2x_{n+1} - x_n) = \sigma((2x_{n+1} - x_n) - 0_V) < (2^p - 1)\epsilon.$$
By p-homogeneity, we thus have
$$\begin{aligned} \forall k \in \mathbb{N}: \quad \sigma\big(2^k x_{N+k} - 2^{k-1} x_{N+k-1}\big) &= \sigma\big(2^{k-1}(2x_{N+k} - x_{N+k-1})\big) \\ &= 2^{(k-1)p} \sigma(2x_{N+k} - x_{N+k-1}) \\ &< 2^{(k-1)p}(2^p - 1)\epsilon. \end{aligned}$$
Next, a telescoping sum in conjunction with the Triangle Inequality yields
$$\forall m \in \mathbb{N}: \quad \sigma(2^m x_{N+m} - x_N) = \sigma\left(\sum_{k=1}^{m} \big(2^k x_{N+k} - 2^{k-1} x_{N+k-1}\big)\right)$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{m} \sigma\big(2^k x_{N+k} - 2^{k-1} x_{N+k-1}\big) \\
&< \sum_{k=1}^{m} 2^{(k-1)p}(2^p - 1)\epsilon \\
&= (2^{mp} - 1)\epsilon.
\end{aligned}
$$

Then by p-homogeneity again,

$$
\begin{aligned}
\forall m \in \mathbb{N}: \quad \sigma\left(x_{N+m} - \frac{1}{2^m} x_N\right) &= \sigma\left(\frac{1}{2^m}(2^m x_{N+m} - x_N)\right) \\
&= \frac{1}{2^{mp}} \sigma(2^m x_{N+m} - x_N) \\
&< \left(1 - \frac{1}{2^{mp}}\right)\epsilon.
\end{aligned}
$$

Applying the Triangle Inequality and p-homogeneity once more, we get

$$
\forall m \in \mathbb{N}: \quad \sigma(x_{N+m}) < \sigma\left(\frac{1}{2^m} x_N\right) + \left(1 - \frac{1}{2^{mp}}\right)\epsilon = \frac{1}{2^{mp}} \sigma(x_N) + \left(1 - \frac{1}{2^{mp}}\right)\epsilon.
$$

Consequently,

$$
\limsup_{n \to \infty} \sigma(x_n) = \limsup_{m \to \infty} \sigma(x_{N+m}) \leq \limsup_{m \to \infty} \left[\frac{1}{2^{mp}} \sigma(x_N) + \left(1 - \frac{1}{2^{mp}}\right)\epsilon\right] = \epsilon.
$$

As $\epsilon > 0$ is arbitrary, we obtain

$$
\lim_{n \to \infty} \sigma(x_n - 0_V) = \lim_{n \to \infty} \sigma(x_n) = 0.
$$

Finally, as $\sigma \in \mathcal{S}$ is arbitrary, (2) of Proposition 2.7 says that $(x_n)_{n \in \mathbb{N}} \xrightarrow{\tau_{\mathcal{S}}} 0_V$. $\square$

By Proposition 2.8, the class of good topological vector spaces includes:

- All locally convex topological vector spaces.
- All L^p-spaces for $p \in (0, 1)$, which are generally not locally convex.

In the next section, we will give an example of a bad topological vector space.

3. A Bad Topological Vector Space from Probability Theory

Before we present the example, let us first fix some probabilistic terminology.

Definition 3.1. Let $(\Omega, \Sigma, \mathsf{P})$ be a probability space.

- A measurable function from (Ω, Σ) to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is called a *random variable*.[1]
- The $\mathbb{R}$-vector space of random variables on (Ω, Σ) is denoted by $\mathsf{RV}(\Omega, \Sigma)$.
- Let $(X_\lambda)_{\lambda \in \Lambda}$ be a net in $\mathsf{RV}(\Omega, \Sigma)$, and let $X \in \mathsf{RV}(\Omega, \Sigma)$. Then $(X_\lambda)_{\lambda \in \Lambda}$ is said to *converge in probability* to X (for P) if and only if for each $\epsilon > 0$, we have

$$
\lim_{\lambda \in \Lambda} \mathsf{P}(\{\omega \in \Omega \mid |X_\lambda(\omega) - X(\omega)| > \epsilon\}) = 0,
$$

  in which case, we write $(X_\lambda)_{\lambda \in \Lambda} \xrightarrow{\mathsf{P}} X$.

The following theorem says that convergence in probability is convergence with respect to a vector-space topology on the vector space of random variables.

[1] $\mathscr{B}(\mathbb{R})$ denotes the Borel σ-algebra generated by the standard topology on $\mathbb{R}$.

Theorem 3.2. *Let* $(\Omega, \Sigma, \mathsf{P})$ *be a probability space, and define a pseudo-metric* ρ_{P} *on* $\mathsf{RV}(\Omega, \Sigma)$ *by*

$$\forall X, Y \in \mathsf{RV}(\Omega, \Sigma) : \quad \rho_{\mathsf{P}}(X, Y) \overset{\text{df}}{=} \int_{\Omega} \frac{|X - Y|}{1 + |X - Y|} \, \mathsf{dP}.$$

Then the topology τ_{P} *on* $\mathsf{RV}(\Omega, \Sigma)$ *generated by* ρ_{P} *has the following properties:*

- τ_{P} *is a vector-space topology.*
- *Let* $(X_\lambda)_{\lambda \in \Lambda}$ *be a net in* $\mathsf{RV}(\Omega, \Sigma)$*. Then for each* $X \in \mathsf{RV}(\Omega, \Sigma)$*, we have*

$$(X_\lambda)_{\lambda \in \Lambda} \xrightarrow{\mathsf{P}} X \qquad \iff \qquad (X_\lambda)_{\lambda \in \Lambda} \xrightarrow{\tau_{\mathsf{P}}} X.$$

Proof. Please refer to Problems 6, 10 and 14 in Section 5.2 of [**1**]. □

Now, for each $k \in \mathbb{N}$, define a probability measure c_k on $([k], \mathcal{P}([k]))$ by

$$\forall A \subseteq [k] : \quad \mathsf{c}_k(A) \overset{\text{df}}{=} \frac{\mathsf{Card}(A)}{k},$$

and let $(\Omega, \Sigma, \mathsf{P})$ denote the product probability space $\displaystyle\prod_{k=1}^{\infty} ([k], \mathcal{P}([k]), \mathsf{c}_k)$. Define a sequence $(S_n)_{n \in \mathbb{N}}$ in Σ by

$$\forall n \in \mathbb{N} : \quad S_n \overset{\text{df}}{=} \left\{ \mathbf{v} \in \prod_{k=1}^{\infty} [k] \;\middle|\; \mathbf{v}(n) = 1 \right\}.$$

Then $\mathsf{P}(S_n) = \dfrac{1}{n}$ for all $n \in \mathbb{N}$, and the S_n's form mutually-independent events.

Next, define a sequence $(Y_n)_{n \in \mathbb{N}}$ in $\mathsf{RV}(\Omega, \Sigma)$ by

$$\forall n \in \mathbb{N} : \quad Y_n \overset{\text{df}}{=} 2^n \chi_{S_n},$$

where χ_{S_n} denotes the indicator function of S_n. Then we get for each $\epsilon > 0$ that

$$\lim_{n \to \infty} \mathsf{P}(\{\omega \in \Omega \mid |Y_n(\omega)| > \epsilon\}) = \lim_{n \to \infty} \mathsf{P}(S_n) = \lim_{n \to \infty} \frac{1}{n} = 0.$$

The first equality is obtained because, for each $\epsilon > 0$, we have $2^n > \epsilon$ for all $n \in \mathbb{N}$ large enough. Consequently, $(Y_n)_{n \in \mathbb{N}} \xrightarrow{\mathsf{P}} 0_{\Omega \to \mathbb{R}}$.

Define a new sequence $(X_n)_{n \in \mathbb{N}}$ in $\mathsf{RV}(\Omega, \Sigma)$ by

$$\forall n \in \mathbb{N} : \quad X_n \overset{\text{df}}{=} \begin{cases} 0_{\Omega \to \mathbb{R}} & \text{if } n = 1; \\ \displaystyle\sum_{k=1}^{n-1} \frac{1}{2^{n-k}} Y_k & \text{if } n \geq 2. \end{cases}$$

Then $2X_2 - X_1 = 2X_2 = Y_1$, and

$$\begin{aligned} \forall n \in \mathbb{N}_{\geq 2} : \quad 2X_{n+1} - X_n &= 2 \sum_{k=1}^{n} \frac{1}{2^{n+1-k}} Y_k - \sum_{k=1}^{n-1} \frac{1}{2^{n-k}} Y_k \\ &= \sum_{k=1}^{n} \frac{1}{2^{n-k}} Y_k - \sum_{k=1}^{n-1} \frac{1}{2^{n-k}} Y_k \\ &= Y_n. \end{aligned}$$

It follows that $(2X_{n+1} - X_n)_{n \in \mathbb{N}} = (Y_n)_{n \in \mathbb{N}} \xrightarrow{\mathsf{P}} 0_{\Omega \to \mathbb{R}}$.

Gathering what we have thus far, observe that

$$\begin{aligned}\forall n \in \mathbb{N}: \quad X_{2n+1} &= \sum_{k=1}^{2n} \frac{1}{2^{2n+1-k}} Y_k \\ &= \sum_{k=1}^{2n} \frac{1}{2^{2n+1-k}} \left(2^k \chi_{S_k}\right) \\ &= \sum_{k=1}^{2n} 2^{2k-2n-1} \chi_{S_k} \\ &\geq \sum_{k=n+1}^{2n} 2^{2k-2n-1} \chi_{S_k} \\ &\geq \sum_{k=n+1}^{2n} \chi_{S_k} \\ &\geq \chi_{\bigcup_{k=n+1}^{2n} S_k}.\end{aligned}$$

As the S_k's are mutually independent, their complements are as well, so

$$\begin{aligned}\forall n \in \mathbb{N}: \quad \mathsf{P}\left(\left\{\omega \in \Omega \;\middle|\; |X_{2n+1}(\omega)| > \frac{1}{2}\right\}\right) &\geq \mathsf{P}\left(\bigcup_{k=n+1}^{2n} S_k\right) \\ &= 1 - \mathsf{P}\left(\Omega \middle\backslash \bigcup_{k=n+1}^{2n} S_k\right) \\ &= 1 - \mathsf{P}\left(\bigcap_{k=n+1}^{2n} \Omega \setminus S_k\right) \\ &= 1 - \prod_{k=n+1}^{2n} \mathsf{P}(\Omega \setminus S_k) \\ &= 1 - \prod_{k=n+1}^{2n} \left(1 - \frac{1}{k}\right) \\ &= 1 - \prod_{k=n+1}^{2n} \frac{k-1}{k} \\ &= 1 - \frac{n}{2n} \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Hence, $(X_n)_{n \in \mathbb{N}}$ does not converge to $0_{\Omega \to \mathbb{R}}$ in probability. By Theorem 3.2:

Proposition 3.3. $(\mathsf{RV}(\Omega, \Sigma), \tau_{\mathsf{P}})$ *is therefore a bad topological vector space.*

By Proposition 2.4, $(X_n)_{n \in \mathbb{N}}$ does not, in fact, converge in probability at all.

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